

ACYLINDRICAL ACTIONS ON TREES AND THE FARRELL–JONES CONJECTURE

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ABSTRACT. We show that for groups acting acylindrically on simplicial trees the K - and L -theoretic Farrell–Jones Conjecture relative to the family of subgroups consisting of virtually cyclic subgroups and all subconjugates of vertex stabilisers holds. As an application, for amalgamated free products acting acylindrically on their Bass–Serre trees we obtain an identification of the associated Waldhausen Nil-groups with a direct sum of Nil-groups associated to certain virtually cyclic groups. This identification generalizes a result by Lafont and Ortiz. For a regular ring and a strictly acylindrical action these Nil-groups vanish. In particular, all our results apply to amalgamated free products over malnormal subgroups.

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INTRODUCTION

The Farrell–Jones Conjecture (for a group G and relative to the family \mathcal{VCYC} of virtually cyclic subgroups) predicts an isomorphism between the K - resp. L -groups of a group ring RG and the value of a certain G -homology theory on a certain classifying space. Such an isomorphism can be seen as a way to (potentially) compute $K_*(RG)$ resp. $L_*^{\langle -\infty \rangle}(RG)$ from smaller ‘building blocks’, namely the K - resp. L -groups of the group rings of the virtually cyclic subgroups of G . The

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conjecture has many implications. For example, if the K -theoretic version of the conjecture holds for a torsion-free group G , then both the Whitehead group $\text{Wh}(G)$ and the reduced projective class group $\tilde{K}_0(\mathbb{Z}G)$ vanish. Also, if both the K - and L -theoretic version hold for a group G , then the Borel Conjecture for G follows. The Farrell–Jones Conjecture has been studied intensively over the past decade and is known for a large class of groups including hyperbolic groups [BLR08a, BL12a], $\text{CAT}(0)$ -groups [BL12a, Weg12], virtually solvable groups [Weg15], $\text{GL}_n(\mathbb{Z})$ [BLRR14] and mapping class groups [BB16]. So far, no counter-example is known.

Proving the Farrell–Jones Conjecture for a group G often starts with considering a suitable action of G on a nice enough space X . Then it can be useful to regard the Farrell–Jones Conjecture not relative to \mathcal{VCYC} , but relative to a (larger) family \mathcal{F} that includes all point-stabilisers of $G \curvearrowright X$: If one shows the conjecture for G relative to \mathcal{F} and all point-stabilisers are known to satisfy the conjecture relative to \mathcal{VCYC} , the Farrell–Jones Conjecture for G relative to \mathcal{VCYC} follows. Recently Bartels used this approach on relatively hyperbolic groups [Bar17], and the first objective of the present paper is to apply the same tactic to groups acting acylindrically on trees.

The precise formulation of and necessary background on the Farrell–Jones conjecture are given in Section 1. A detailed exposition on the numerous application of the Farrell–Jones Conjecture can be found in [BLR08b], [LR05], [Lüc10]. A survey on the various methods used to prove the conjecture in various cases can be found in [Bar14].

A discrete group G acts *k-acylindrically* on a simplicial tree T if the pointwise stabiliser of every geodesic segment of length k in T is finite (we do not require these stabilisers to be trivial or to have uniform cardinality). An action $G \curvearrowright T$ is called *acylindrical* if it is *k-acylindrical* for some k . Examples of acylindrical actions can be given by considering an amalgamated free product $G = A *_C B$, where C is an almost malnormal subgroup in A or B .

In general, for a group action $G \curvearrowright T$ on a simplicial tree, denote by \mathcal{F}_T the family of subgroups of G which are subconjugate to vertex stabilisers, and by \mathcal{F}_∂ the family of subgroups of G that fix a pair of boundary points of T pointwise.

Theorem A. *Let G be a group acting acylindrically on a simplicial tree T and let the family $\mathcal{F} := \mathcal{F}_T \cup \mathcal{F}_\partial$ be as above. Let \mathcal{F}_2 be the family of subgroups of G that contain a group in \mathcal{F} as a subgroup of index ≤ 2 . Then*

- a) G satisfies the K -theoretic Farrell–Jones Conjecture relative to \mathcal{F} ;*
- b) G satisfies the L -theoretic Farrell–Jones Conjecture relative to \mathcal{F}_2 .*

In the special cases of 0- or 1-acylindrical actions this theorem was already known: If G acts 0-acylindrically on a tree T (and G is finitely generated), then G is hyperbolic and the Farrell–Jones Conjecture relative to $\mathcal{VCYC} \subset \mathcal{F}_T \cup \mathcal{F}_\partial$ holds [BLR08a, BL12a]. A more recent result of Bartels [Bar17] for relatively hyperbolic groups covers, in particular, the 1-acylindrical case. We will see in Section 3 that it is easy to construct actions that are acylindrical but not 1-acylindrical.

If the vertex stabilisers of the action $G \curvearrowright T$ are known to satisfy the Farrell–Jones Conjecture relative to \mathcal{VCYC} , we obtain the following corollary and its L -theoretic analogue, which is formulated in Corollary 4.3.

Corollary 4.2. *Let G be a group acting acylindrically on a simplicial tree T . If all vertex stabilisers of $G \curvearrowright T$ satisfy the K -theoretic Farrell–Jones Conjecture relative to \mathcal{VCYC} , then G satisfies the K -theoretic Farrell–Jones Conjecture relative to \mathcal{VCYC} .*

The Farrell–Jones Conjecture (with coefficients in additive categories) relative to \mathcal{VCYC} has various useful inheritance properties. For example, if two groups A and B satisfy the conjecture, then so do $A \times B$ and $A * B$. If A and B are abelian, then an amalgamated free product $G = A *_C B$ satisfies the Farrell–Jones Conjecture relative to \mathcal{VCYC} by [GMR15], but for a general amalgamated free product this inheritance property is not known. In this context the last corollary can also be interpreted as follows: Any amalgamated free product $G = A *_C B$ with C being almost malnormal in either A or B acts acylindrically on its Bass–Serre tree. Thus, the above corollary implies that the class of groups satisfying the Farrell–Jones Conjecture relative to \mathcal{VCYC} is closed under taking amalgamated free products over an almost malnormal subgroup.

Another angle on amalgamated free products and the computation of algebraic K -theory is the following: For a group $G = A *_C B$ one can ask whether there is a Mayer–Vietoris type exact sequence

$$\dots \rightarrow K_n(RC) \rightarrow K_n(RA) \oplus K_n(RB) \rightarrow K_n(RG) \rightarrow K_{n-1}(RC) \rightarrow \dots$$

In general this is not the case, and Waldhausen famously introduced an exact category $\mathfrak{Nil}(RC; R[A - C], R[B - C])$ encoding to what extent exactness of the above sequence fails in [Wal78a, Wal78b]. In particular, *Waldhausen Nil-groups* $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ associated to the amalgamated free product $G = A *_C B$ are direct summands in the homotopy groups of a certain non-connective spectrum version of $\mathfrak{Nil}(RC; R[A - C], R[B - C])$, and the sequence

$$\dots \rightarrow K_n(RC) \rightarrow K_n(RA) \oplus K_n(RB) \rightarrow K_n(RG)/\widetilde{\text{Nil}}_{n-1} \rightarrow K_{n-1}(RC) \rightarrow \dots$$

becomes exact. Naturally, one is interested in identifying $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ with something that is known to vanish often. For a precise, yet concise definition of the groups $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ we refer the reader to [DQR11, Section 3].

For an amalgamated free product $G = A *_C B$ that acts acylindrically on the associated Bass–Serre tree *and* is already known to satisfy the Farrell–Jones Conjecture relative to \mathcal{VCYC} , Lafont and Ortiz obtained an identification in [LO09]. Building on Theorem A, we are able to lift their second assumption. Namely, we obtain the following generalisation of their theorem.

Theorem B. *Let $G = A *_C B$ act acylindrically on the associated Bass–Serre tree T . Let \mathcal{L} be a set of representatives for the orbits of the action $G \curvearrowright \partial T \times \partial T \setminus \text{diag}$. Then for any ring R there are isomorphisms*

$$\begin{aligned} & \widetilde{\text{Nil}}_{n-1}(RC; R[A - C], R[B - C]) \\ & \cong \bigoplus_{L \in \mathcal{L}} \text{coker}(H_n^{G_L}(E_{\mathcal{FTN}} G_L; \mathbf{K}_R) \rightarrow H_n^{G_L}(\text{pt}; \mathbf{K}_R)) \end{aligned}$$

for $n \in \mathbb{Z}$.

More information on the right hand side of the above isomorphisms—in particular, when they are known to vanish—can be found in Section 5.

The structure of this paper is as follows: Section 1 contains the background on the Farrell–Jones Conjecture necessary for us. In particular, we review the method to prove the Farrell–Jones Conjecture for a group G by establishing the existence of a suitable compact space X and intricate G -equivariant covers of $G \times X$. In our case of a group acting on a tree T , the space X will be the Bowditch compactification of T and this is the content of Section 2. Section 3 then gives some context on groups acting acylindrically on trees. To make the paper more accessible to readers less familiar with the covering-construction-culture around the Farrell–Jones Conjecture, Section 4 contains the proof of Theorem A modulo the lengthy and technical part and Section 5 contains the application to Waldhausen Nil-groups. Afterwards, the rest of the paper describes the plan to construct suitable covers of $G \times X$ in Section 6 and carries it out in Sections 7–10.

Conventions. Any countable group in this paper is assumed to be equipped with an implicitly chosen proper (left-)invariant metric. Given a metric space (X, d) , for $r > 0$ and $x \in X$, we denote—if not explicitly stated otherwise—by $B_r^d(x) = B_r(x)$ the *closed* ball of radius r around x . By a generalised metric on X we mean a function $d : X \times X \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ which is symmetric, satisfies the triangle inequality and $d(x, y) = 0$ if and only if $x = y$. Finally, we stress that it is hard to draw an unbounded geodesic ray or a tree which is not locally finite. So all figures in this paper have to be regarded as ‘incomplete illustrations’ that hopefully still capture the essence of the notion or proof at hand.

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1. BACKGROUND ON THE FARRELL–JONES CONJECTURE

The conjecture goes back to Farrell and Jones [FJ93], and since then, its formulation has been further conceptualised and broadly generalised. Its present form (with coefficients in additive categories) was introduced by Bartels and Reich [BR07] (in the K -theory case) and Bartels and Lück [BL10] (in the L -theory case). The existence of an axiomatic formulation of geometric conditions implying the Farrell–Jones Conjecture for a group G , which was also introduced by Bartels, Lück and Reich (cf. Section 1.1), allows us—for almost all of this paper—to display (or feign) ignorance about the precise nature of most objects appearing in the Farrell–Jones Conjecture. Thus, we restrict ourselves to stating the conjecture (as formulated in [BR07] and [BL10]) essentially without further comment on the objects involved.

A *family \mathcal{F} of subgroups of G* is a collection of subgroups of G closed under conjugation and taking subgroups. Examples being the family \mathcal{ALL} of all subgroups of G , the family \mathcal{FIN} of all finite subgroups of G , and most importantly, the family \mathcal{VCC} of all virtually cyclic subgroups of G .

Definition 1.1. (*K -theoretic Farrell–Jones Conjecture relative to \mathcal{F}*) A group G satisfies the *K -theoretic Farrell–Jones Conjecture relative to the family of subgroups \mathcal{F}* if for all additive G -categories \mathcal{A} the K -theoretic assembly map

$$\mathrm{asmb}_n(G, \mathcal{F}, \mathcal{A}) : H_n^G(E_{\mathcal{F}}G; \mathbf{K}_{\mathcal{A}}) \rightarrow H_n^G(\mathrm{pt}; \mathbf{K}_{\mathcal{A}}) \cong K_n \left(\int_G \mathcal{A} \right),$$

induced by the projection $E_{\mathcal{F}}G \rightarrow \mathrm{pt}$, is an isomorphism for all $n \in \mathbb{Z}$.

Definition 1.2. (*L-theoretic Farrell–Jones Conjecture relative to \mathcal{F}*) A group G satisfies the *L-theoretic Farrell–Jones Conjecture relative to the family of subgroups \mathcal{F}* if for all additive G -categories \mathcal{A} with involution the *L-theoretic assembly map*

$$\mathrm{asmb}_n(G, \mathcal{F}, \mathcal{A}) : H_n^G(E_{\mathcal{F}}G; \mathbf{L}_{\mathcal{A}}^{-\infty}) \rightarrow H_n^G(\mathrm{pt}; \mathbf{L}_{\mathcal{A}}^{-\infty}) \cong L_n^{\langle -\infty \rangle} \left(\int_G \mathcal{A} \right),$$

induced by the projection $E_{\mathcal{F}}G \rightarrow \mathrm{pt}$, is an isomorphism for all $n \in \mathbb{Z}$.

As of now we use the term *Farrell–Jones Conjecture (relative to \mathcal{F})* when talking about both of the above definitions simultaneously. So in particular, and contrary to the existing literature, we do *not* take $\mathcal{VC}\mathcal{C}$ to be the default family.

1.1. A geometric condition implying the Farrell–Jones Conjecture. Along with the proof of the *K-theoretic Farrell–Jones Conjecture* for hyperbolic groups, Bartels, Lück and Reich showed that the existence of a particular group action $G \curvearrowright X$ admitting intricate G -invariant covers implies the Farrell–Jones Conjecture (cf. [BLR08a, Theorem 1.1]). Since then these geometric conditions have been reformulated, generalised and adapted in several ways (cf. [BL12a, Section 1], [Weg12, Theorem 1.1 and Sections 2 and 3], [Bar17, Definition 0.1 and Theorem 4.3]). We review here the formulation that suits us most while mainly following [Bar17].

Definition 1.3. Let \mathcal{F} be a family of subgroups of a group G . A subset $U \subset X$ of a G -space X is called an *\mathcal{F} -subset* if there is $F \in \mathcal{F}$ such that $gU = U$ for $g \in F$ and $gU \cap U = \emptyset$ for $g \notin F$. A collection \mathcal{U} of subsets of X is said to be *G -invariant* if $gU \in \mathcal{U}$ for all $g \in G$ and $U \in \mathcal{U}$. An (*open*) *\mathcal{F} -cover* of X is a cover of X consisting of (*open*) \mathcal{F} -subsets.

Definition 1.4. Let \mathcal{U} be a collection of subsets of some space X . The *order* of \mathcal{U} is $\leq N$ if each $x \in X$ is contained in at most $N + 1$ members of \mathcal{U} . If the collection \mathcal{U} is a cover of X , the order of \mathcal{U} will also be called the *dimension* of \mathcal{U} and then is denoted by $\dim \mathcal{U}$.

For the connection of the next definition to amenable actions we refer the reader to [Bar17, Remark 0.4].

Definition 1.5. (cf. [Bar17, Definition 0.1]) Let G be a countable group (with a chosen proper (left-)invariant metric) and let \mathcal{F} be a family of subgroups. An action of G on a space X is *$N - \mathcal{F}$ -amenable* if for any $\alpha > 0$ there exists an open G -invariant \mathcal{F} -cover \mathcal{U}_{α} of $G \times X$ (equipped with the diagonal action) with the following properties:

- a) the dimension of \mathcal{U}_{α} is at most N ;
- b) for all $(g, x) \in G \times X$ there is $U \in \mathcal{U}_{\alpha}$ with $B_{\alpha}(g) \times \{x\} \subset U$.

An action is *finitely \mathcal{F} -amenable* if it is $N - \mathcal{F}$ -amenable for some N .

Since we will not construct those types of covers in one go, we introduce the following useful terminology reminiscent of [BLR08a, Assumption 1.4].

Definition 1.6. Let $\alpha > 0$ be given. A collection \mathcal{U} of subsets of $G \times X$ is *wide for Y* , where $Y \subset G \times X$ is a subset, if for all $(g, x) \in Y$ there is $U \in \mathcal{U}$ with $B_{\alpha}(g) \times \{x\} \subset U$. The collection \mathcal{U} is *wide* if it is wide for $Y = G \times X$.

Definition 1.7. ([BL12a, Definition 1.5]) Let X be a metric space and $N \in \mathbb{N}$. Then X is *controlled N -dominated* if, for every $\epsilon > 0$, there are a finite CW-complex K_{ϵ} of dimension at most N , maps $i : X \rightarrow K_{\epsilon}$, $p : K_{\epsilon} \rightarrow X$ and a homotopy $H : X \times [0, 1] \rightarrow X$ between $p \circ i$ and id_X such that for every $x \in X$ the diameter

of $\{H(x, t) \mid t \in [0, 1]\}$ is at most ϵ . X is *controlled finitely-dominated* if X is controlled N -dominated for some N .

Theorem 1.8. (cf. [Bar17, Theorem 4.3]) *Let G be a group and \mathcal{F} be a family of subgroups. Let \mathcal{F}_2 be the family of subgroups of G that contain a group in \mathcal{F} as a subgroup of index ≤ 2 . If G admits a finitely \mathcal{F} -amenable action on a compact contractible controlled finitely-dominated metric space X , then*

- a) G satisfies the K -theoretic Farrell–Jones Conjecture relative to \mathcal{F} ;
- b) G satisfies the L -theoretic Farrell–Jones Conjecture relative to \mathcal{F}_2 .

Proof. b) is the conclusion of [BL12a, Theorem 1.1 (ii)]. The assumptions of [BL12a, Theorem 1.1] were phrased as ‘ G is transfer reducible over \mathcal{F} ’ (cf. [BL12a, Definition 1.8]). The notion of transfer reducibility is more general than $N - \mathcal{F}$ -amenability of an action on a suitable space. (Transfer reducibility neither demands a strict action of G on X , nor that X is independent of the given $\alpha > 0$.) Moreover, transfer reducibility demands G -invariant covers for the action of G on $G \times X$ that is given by $h(g, x) := (hg, x)$. Nevertheless, using that $(g, x) \mapsto (g, g^{-1}x)$ is a G -invariant homeomorphism from $G \times X$ with the diagonal action to $G \times X$ with the action described above, it is straightforward to check from the respective definitions that the existence of a finitely \mathcal{F} -amenable action of G on a compact contractible controlled finitely-dominated metric space X implies that G is transfer reducible over \mathcal{F} (cf. the comment behind [BL12a, Definition 1.4]).

a) is the conclusion of [Weg12, Theorem 1.1]. The assumptions of [Weg12, Theorem 1.1] were phrased as ‘ G is strongly transfer reducible over \mathcal{F} ’ (cf. [Weg12, Definition 3.1]). Again, this notion is more general than $N - \mathcal{F}$ -amenability and one can check, by translation of terminology, that the existence of a finitely \mathcal{F} -amenable action of G on a compact contractible controlled finitely-dominated metric space X implies that G is strongly transfer reducible over \mathcal{F} . \square

1.2. The Farrell–Jones Conjecture and directed colimits. The Farrell–Jones Conjecture described above has various useful inheritance properties, see for instance [BEL08, Section 5] and [BFL14, Section 2.3]. These sources have mostly been interested in the conjecture relative to the family \mathcal{VCYC} , and subsequently the author could not find the following lemma in the literature.

Lemma 1.9. *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps). Let $G = \operatorname{colim}_{i \in I} G_i$, with structure maps $\phi_i : G_i \rightarrow G$, and let \mathcal{F} be a family of subgroups of G . If for all $i \in I$ the group G_i satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to $\phi_i^* \mathcal{F} := \{H \leq G_i \mid \phi_i(H) \in \mathcal{F}\}$, then G satisfies the K -theoretic (resp. L -theoretic) Farrell–Jones Conjecture relative to \mathcal{F} .*

Proof. In the L -theoretic case, if $\mathcal{F} = \mathcal{VCYC}$ and $\phi_i^* \mathcal{F}$ is replaced by \mathcal{VCYC} as well ($\mathcal{VCYC}(G_i)$ that is), then this lemma is [BL10, Corollary 0.8]. This corollary is indicated in [BL10] to come from [BEL08, Theorem 5.6], which is formulated generally enough to also conclude the lemma in the K - and L -theoretic case provided the families \mathcal{F} and $\phi_i^* \mathcal{F}$ of subgroups come from a class of groups closed under isomorphisms and taking quotients—a restriction that is not acceptable for our purposes. Luckily, using [BEL08, Theorem 5.2] instead, in combination with a straightforward generalisation of [BL10, Theorem 0.7] to arbitrary families, allows

to deduce [BL10, Corollary 0.8] from [BEL08] in a way that works verbatim for arbitrary families and the K -theoretic case as well. \square

We wish to employ Theorem 1.8 in order to prove Theorem A. The next section introduces the right compact contractible controlled finitely-dominated metric space for this.

2. THE OBSERVERS' TOPOLOGY ON A TREE

For proper geodesic metric hyperbolic spaces (so in particular for a locally finite tree T) adding the Gromov boundary results in a compact metrisable space (see [BH99, III.H.3, in particular 3.7, 3.18 (4)]). Since we are interested mostly in trees that are not locally finite, changing the topology on T itself becomes necessary.

We denote by ∂T the geodesic/Gromov boundary of T , by $V(T)$ the set of vertices of T , by $V_0(T)$ the set of vertices of finite degree/valency of T and set $V_\infty(T) := V(T) \setminus V_0(T)$. Furthermore, d_T will always denote the path-metric on T .

Notation 2.1. For two points x, y in $T \cup \partial T$ we denote both the geodesic from x to y and its image by $[x, y]$. In particular, our terminology does not distinguish between geodesics, geodesic rays and bi-infinite geodesic rays. Moreover, we use $(x, y]$ and $[x, y)$ for $[x, y] \setminus \{x\}$ and $[x, y] \setminus \{y\}$, respectively (provided $x \in T$ and $y \in T$, respectively).

Bowditch's construction of the observers' topology amounts to compactifying $\Delta(T) := V_\infty(T) \cup \partial T$. We will need to use a compactification—via the observers' topology—of all of $T \cup \partial T$ instead. Written in the more general language of \mathbb{R} -trees this can be found in [CHL07, Section 1].

Definition 2.2. The topology on $T \cup \partial T$ given by the basis

$$\{M(z, A) \mid z \in T \cup \partial T, A \subset T \text{ finite}\},$$

where $M(z, A) := \{y \in T \cup \partial T \mid A \cap [z, y] = \emptyset\}$, is called the *observers' topology*. The resulting space is the *Bowditch compactification* of T and will be denoted by \bar{T}^{obs} . The *Bowditch boundary* of T is the subspace $\Delta(T) = V_\infty(T) \cup \partial T \subset \bar{T}^{obs}$. We further set $\Delta_+(T) := V(T) \cup \partial T \subset \bar{T}^{obs}$.

The above naming is justified since it is compatible with Bowditch's construction, i.e. the subspace topology on $\Delta(T) \subset \bar{T}^{obs}$ is the topology constructed by Bowditch in [Bow12, Section 8, p. 51ff]. Note also that $M(z, A)$ consists of all points in $T \cup \partial T$ that can be reached from z via a geodesic that does not run through a point of A . Therefore, if $x \in M(z, A)$, we have $M(x, A) = M(z, A)$.

Lemma 2.3. [CHL07, Proposition 1.13] *The space \bar{T}^{obs} is Hausdorff. Moreover, if T is separable, then the space \bar{T}^{obs} is separable and compact.*

Lemma 2.4. *If T is countable (i.e. has only countably many edges), then \bar{T}^{obs} is second-countable and metrisable. In particular, (any subspace of) \bar{T}^{obs} is separable.*

Proof. If T is countable, then T is separable. Choose a countable dense subset Q of T which contains all vertices of T . Then

$$\{M(z, A) \mid z \in Q, A \subset Q \text{ finite}\}$$

is countable and still a basis for the observers' topology, as $M(x, A) = M(z, A)$ holds for all $z \in M(x, A)$. So \bar{T}^{obs} is second-countable. Since, by the above lemma,

\bar{T}^{obs} is also compact Hausdorff, the space \bar{T}^{obs} is metrisable. Finally, for metric spaces being second-countable and being separable are equivalent properties, so any subspace of \bar{T}^{obs} is separable as well. \square

We proceed to show that \bar{T}^{obs} is contractible and controlled 1-dominated (see Definition 1.7). We split the proof into two lemmata.

Lemma 2.5. *Let d be a metric on \bar{T}^{obs} compatible with the observers' topology. For every $\epsilon > 0$ there is a finite subtree $K_\epsilon \subset T$ and a continuous projection $p : \bar{T}^{obs} \rightarrow K_\epsilon$ such that $d(x, p(x)) < \epsilon$ for all $x \in \bar{T}^{obs}$.*

Proof. For this proof we denote the open ball of radius r around $x \in \bar{T}^{obs}$ with respect to d by $B_r(x)$. Certainly,

$$\{B_{\epsilon/2}(x) \mid x \in \bar{T}^{obs}\}$$

is an open cover of \bar{T}^{obs} . Since the $M(x, A)$ with $x \in T \cup \partial T$ and A a finite subset of T form an open neighbourhood basis at x for the observers' topology, there is for each $B_{\epsilon/2}(x)$ an open set $M(x, A_x) \subset B_{\epsilon/2}(x)$. So the collection

$$\{M(x, A_x) \mid x \in \bar{T}^{obs}\}$$

is an open cover of \bar{T}^{obs} as well. Since \bar{T}^{obs} is compact, we can find a finite subcover $M(x_1, A_1), \dots, M(x_l, A_l)$. Without loss of generality we can assume that the x_i lie in $V(T)$. Let K_ϵ be the finite subtree of T spanned by x_1, \dots, x_l . We define $p : \bar{T}^{obs} \rightarrow K_\epsilon$ as follows: Fix some point $b \in K_\epsilon$ and define $p : \bar{T}^{obs} \rightarrow K_\epsilon$ by sending x to the last vertex on $[b, x]$ that is contained in K_ϵ .

To show $d(x, p(x)) < \epsilon$, we argue as follows: For $x \in K_\epsilon$ we have $d(x, p(x)) = 0$, since $p|_{K_\epsilon} = \text{id}_{K_\epsilon}$. For any point $x \notin K_\epsilon$ there is an x_{i_x} such that $x \in M(x_{i_x}, A_i)$. Since K_ϵ is the tree spanned by x_1, \dots, x_l , the point $p(x)$ lies on $[x, x_{i_x}]$, and hence $p(x) \in M(x_{i_x}, A_i)$. Therefore, $d(p(x), x) < \epsilon$. The proof that p is indeed continuous is not very illuminating and only uses the fact that inside each $B_\delta(x)$ one can find suitable $M(x, A)$ that are guaranteed to be mapped into the given $B_\epsilon(p(x))$. \square

Lemma 2.6. *The space \bar{T}^{obs} is controlled 1-dominated (by the K_ϵ of Lemma 2.5). In particular, \bar{T}^{obs} is contractible.*

Proof. Let d be a metric on \bar{T}^{obs} compatible with the observers' topology. Again, for this proof denote the open ball of radius r around $x \in \bar{T}^{obs}$ with respect to d by $B_r(x)$. Let $\epsilon > 0$ and let $p : \bar{T}^{obs} \rightarrow K_\epsilon$ be as in the previous lemma, i.e. we have a finite subtree K_ϵ of T , fixed a point $b \in K_\epsilon$ and defined $p(x)$ as the last vertex on $[b, x]$ that also lies in K_ϵ . We now have to show that there is a homotopy $H : \bar{T}^{obs} \times [0, 1] \rightarrow \bar{T}^{obs}$ between $\text{id}_{\bar{T}^{obs}}$ and $i \circ p : \bar{T}^{obs} \rightarrow K_\epsilon \rightarrow \bar{T}^{obs}$, where $i : K_\epsilon \rightarrow \bar{T}^{obs}$ is the inclusion, such that the set $\{H(x, t) \mid t \in I\}$ has diameter $\leq 2\epsilon$ for all $x \in \bar{T}^{obs}$.

For $x \in \bar{T}^{obs}$ let $\gamma_{p(x), x} : [0, \infty) \rightarrow T$ be the unique generalised geodesic in T from $p(x)$ to x , i.e. if $x \in T$, then $\gamma(t)$ is stationary for $t \geq d_T(p(x), x)$. We define

$$H(x, t) = \gamma_{p(x), x} \left(\frac{t}{1-t} \right)$$

where for $t = 1$ and $x \in T$ this is to be read as $H(x, 1) = x$, and for $t = 1$ and $x \in \partial T$ this is to be read as $H(x, 1) = [\gamma_{p(x), x}]$. So $H(-, 0) = p$ and $H(-, 1) = id_{\overline{T}^{obs}}$. Furthermore, note that the set $\{H(x, t) \mid t \in I\}$ is exactly the image of the generalised geodesic $\gamma_{p(x), x}$. Any point on $\gamma_{p(x), x}$ is mapped by p to $p(x)$. Thus, this set has diameter $\leq 2\epsilon$ by the previous lemma. Again the proof that H is indeed continuous is an exercise in choosing suitable $M(-, A)$ inside the appropriate $B_\delta(x)$. \square

We will also need to know that the (small inductive) dimension¹ of $\Delta_+(T)$ is 0.

Definition 2.7. Let X be a regular space. The *small inductive dimension* $\text{ind } X$ is $\leq n$ if for all $x \in X$ and all neighbourhoods $V \subset X$ of x there is an open set $U \subset X$ such that $x \in U \subset V$ and $\text{ind } \partial U \leq n - 1$. The small inductive dimension of the empty set is defined as $\text{ind } \emptyset := -1$.

Lemma 2.8. *The space $\Delta_+(T)$ has (small inductive) dimension 0.*

Proof. By definition, $\mathcal{B} := \{M(z, A) \mid z \in T \cup \partial T, A \subset T \text{ finite}\}$ is a basis for the topology of \overline{T}^{obs} . So if V is a neighbourhood of some y in $\Delta_+(T)$, then there is a set $M(z, A) \in \mathcal{B}$ with $y \in M(z, A) \cap \Delta_+(T) =: U \subset V$. In particular U is open in $\Delta_+(T)$ and we wish to show that the boundary of U (in $\Delta_+(T)$) is empty.

Since $y \in M(z, A)$, we can write $U = M(y, A) \cap \Delta_+(T)$. Now let $x_n \rightarrow x$ be a convergent sequence in $\Delta_+(T)$ with $x_n \in U$ for all $n \in \mathbb{N}$. If the x_n are contained in a finite subtree S of T , then—since $x_n \in \Delta_+(T)$ —the sequence must have a constant subsequence and $x \in U$ follows. So assume from now on that the sequence is not contained in a finite subtree of T . If $x \notin U$, then there must be an $a \in A$ lying in the interior of $[z, x]$. Thus $M(x, a) \cap \Delta_+(T)$ is an open set containing x , but none of the x_n . A contradiction to $x_n \rightarrow x$. \square

3. ACYLINDRICAL ACTIONS ON TREES

The original definition of a k -acylindrical action of a group on a simplicial tree is due to Sela [Sel97, p. 528]. And in recent years Osin established Bowditch's more general notion of an acylindrical action of a group on a path-metric space [Bow08, p. 284] as part of his definition of acylindrically hyperbolic groups [Osi16]. We will however use Delzant's definition of a (k, \mathcal{FLN}) -acylindrical action (see [Del99, Definition, p. 1215]), which in the case of simplicial actions on trees generalises both Sela's and Bowditch's definition.

Definition 3.1. Let $k \geq 0$. An action $G \curvearrowright T$ on a simplicial tree is *k -acylindrical* if the pointwise stabiliser of any geodesic segment of length k is finite.

We say the action is *uniformly k -acylindrical*, if there is a uniform bound on the cardinality of the pointwise stabiliser of geodesic segments of length k . The action is *strictly k -acylindrical*, if the pointwise stabiliser of any geodesic segment of length k is trivial. And we say an action is *(strictly/uniformly) acylindrical*, if it is (strictly/uniformly) k -acylindrical for some $k \in \mathbb{N}$.

Remark 3.2. In Sela's original definition of k -acylindricity the action was required to be strictly $k+1$ -acylindrical in the above sense. And Bowditch's definition applied to a simplicial tree is equivalent to a uniformly k -acylindrical action in the above sense.

¹Recall that for separable metrisable spaces, the small inductive dimension coincides with the covering dimension (and the strong inductive dimension) [Nag65, Theorem IV.1, p. 90].

In the case $k = 0$ the action $G \curvearrowright T$ has finite point stabilisers. Hence, if G is finitely generated, then the existence of a 0-acylindrical action on a simplicial tree implies that G is word hyperbolic (since G is finitely generated we can assume without loss of generality that G acts cocompactly on T).

In the case $k = 1$ the action $G \curvearrowright T$ has finite edge stabilisers. Since any tree is, in particular, a fine hyperbolic graph in the sense of Bowditch (cf. [Bow08, Definition after Proposition 2.1, p. 11]), if G is finitely generated the existence of a 1-acylindrical action on a simplicial tree with finitely generated vertex stabilisers implies that G is strongly relatively hyperbolic, i.e. relatively hyperbolic in the sense of Bowditch (cf. [Bow08, Definition 2]). Examples of 1-acylindrical actions on simplicial trees can be readily given: For groups A and B with an (up to isomorphism) common finite subgroup C , the amalgamated free product $A *_C B$ acts 1-acylindrically on the Bass-Serre tree associated to $A *_C B$.

Plenty of examples for k -acylindrical actions that are not 1-acylindrical can be given using the classical notion of an (almost) malnormal subgroup.

Definition 3.3. A subgroup $H \leq G$ is *malnormal* resp. *almost malnormal* if for all $g \in G \setminus H$ the intersection $H \cap gHg^{-1}$ is trivial resp. finite. We call a subgroup $H \leq G$ *uniformly almost malnormal* if there is a bound $K > 0$ such that for all $g \in G \setminus H$ the intersection $H \cap gHg^{-1}$ has at most K elements.

From the explicit description of the Bass-Serre tree associated to an amalgamated free product (see [Ser80, p. 33]) the following lemma follows readily.

Lemma 3.4. (cf. [Sel97, p. 528]) *Let $C \leq A$ be (uniformly) almost malnormal and let $C \leq B$. Then the action of $G := A *_C B$ on the Bass-Serre tree associated to G is (uniformly) 3-acylindrical. In particular, if C is infinite, then G does not act 1-acylindrically on its Bass-Serre tree.*

Similar statements can be formulated for HNN-extensions and graphs of groups using the notion of an almost malnormal collection of subgroups.

There are plenty of examples for (almost) malnormal subgroups (for an overview on results giving such groups see for instance [dlHW14]). In particular, peripheral subgroups in relatively hyperbolic groups are uniformly almost malnormal (see for instance [DS08, Lemma 4.20]). On the other hand, it is worth noting that not all acylindrical actions arise in this fashion from (almost) malnormal subgroups. As can be seen (with some translation of terminology) from the type of example given in [KS71, p. 946 and p. 951], there are amalgamated free products $A *_C B$ that act 3-acylindrically on their associated Bass-Serre tree, with C not being almost malnormal in A or B .

Notation 3.5. Let $G \curvearrowright T$ be an action of a group on a tree. We denote by \mathcal{F}_T the family of subgroups $\mathcal{F}_T := \{H \leq G \mid \exists x \in T : H \leq G_x\}$ and by \mathcal{F}_∂ the family of subgroups $\mathcal{F}_\partial := \{H \leq G \mid \exists (x, y) \in \partial T \times \partial T \setminus \text{diag} : H \leq G_x \cap G_y\}$. Here G_x denotes the stabiliser of a point $x \in T \cup \partial T$.

It is known that $\mathcal{F}_\partial \subset \mathcal{VCVC}$ holds for acylindrical amalgamations (see [LO09, Claim 2]). The author expects the next lemma also to be well-known, but could not find a reference for this particular statement either.

Lemma 3.6. *Let G be a group acting acylindrically on a simplicial tree T . Then each element in \mathcal{F}_∂ is either finite or virtually cyclic of type I (i.e. surjects onto \mathbb{Z}).*

Proof. For $g \in G$, if the translation length $\|g\|$ of g is > 0 , denote by C_g its translation axis. If $\|g\| = 0$, denote by C_g the subtree of T which is pointwise fixed by g . Let $\xi \in \partial T$. Then for all $g \in G_\xi$ the set C_g has unbounded intersection with each geodesic ray ρ representing ξ . ($C_g \cap \rho$ being empty or bounded immediately gives a contradiction to $g \in G_\xi$, since T is a tree; cf. Figure 1 for an exemplary case).

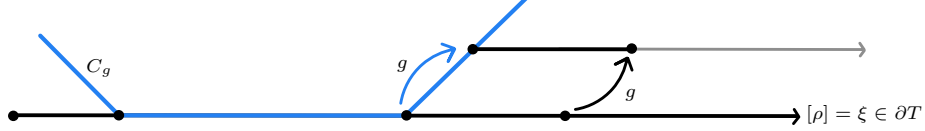


FIGURE 1. Bounded intersection of C_g and ρ implies $g \notin G_\xi$.

Let $(\xi_1, \xi_2) \in \partial T \times \partial T \setminus \text{diag}$. Then any $g \in G_{\xi_1} \cap G_{\xi_2}$ must satisfy $[\xi_1, \xi_2] \subset C_g$ (and equality holds if $\|g\| > 0$). Hence, $G_{\xi_1} \cap G_{\xi_2}$ can only contain finitely many elements with translation length 0. In particular, if the group $G_{\xi_1} \cap G_{\xi_2}$ consists only of elements with translation length 0, this group is finite. So from now on we assume that there is at least one element $\bar{g} \in G_{\xi_1} \cap G_{\xi_2}$ with $\|\bar{g}\| > 0$. Since $\|g\| \in \mathbb{N}$ there is $g_0 \in G_{\xi_1} \cap G_{\xi_2}$ with $\|g_0\| > 0$ minimal.

The translation length $\|h\|$ of any $h \in G_{\xi_1} \cap G_{\xi_2}$ with $\|h\| > 0$ must be divisible by $\|g_0\|$: Assume $\|g_0\|$ does not divide $\|h\|$. Then there are integers α, β such that $\|g_0\| > \gcd(\|g_0\|, \|h\|) = \alpha\|g_0\| + \beta\|h\|$. Hence, the element $g_0^\alpha h^\beta$ acts as a shift on $[\xi_1, \xi_2]$ and therefore lies in $G_{\xi_1} \cap G_{\xi_2}$. Since $g_0^\alpha h^\beta$ has translation length less than $\|g_0\|$, this is a contradiction.

Let $\{g_1, \dots, g_K\}$ be the set of all elements in $G_{\xi_1} \cap G_{\xi_2}$ with translation length 0. If $h \in G_{\xi_1} \cap G_{\xi_2}$ and $\|h\| = 0$, then $h = g_i$ for some $1 \leq i \leq K$ and, in particular, $h = g_i g_0^0 \in g_i \langle g_0 \rangle$. If $h \in G_{\xi_1} \cap G_{\xi_2}$ with $\|h\| = \alpha\|g_0\| > 0$, then one of the two elements $h g_0^{\pm\alpha}$ fixes $[\xi_1, \xi_2]$ pointwise and thus must lie in $\{g_1, \dots, g_K\}$. In other words, we have $h = g_i g_0^\alpha$ for some $\alpha \in \mathbb{Z}$ and $1 \leq i \leq K$. Hence, $\mathbb{Z} = \langle g_0 \rangle$ is of finite index in $G_{\xi_1} \cap G_{\xi_2}$. Moreover, sending $g_i g_0^\alpha$ to g_0^α gives the desired surjection onto \mathbb{Z} . \square

4. THE PROOF OF THEOREM A MODULO \mathcal{F} -AMENABILITY OF $G \curvearrowright \Delta(T)$

Theorem A. *Let G be a group acting acylindrically on a simplicial tree T and let the family $\mathcal{F} := \mathcal{F}_T \cup \mathcal{F}_\partial$ be as in Notation 3.5. Let \mathcal{F}_2 be the family of subgroups of G that contain a group in \mathcal{F} as a subgroup of index ≤ 2 . Then*

- a) G satisfies the K -theoretic Farrell-Jones Conjecture relative to \mathcal{F} ;
- b) G satisfies the L -theoretic Farrell-Jones Conjecture relative to \mathcal{F}_2 .

We would like to use the geometric conditions from Section 1.1 to conclude the above theorem by basically showing that G acts finitely \mathcal{F} -amenable on $\overline{T}^{\text{obs}}$. The major work, which also turns out to be lengthy and technical, towards this is to establish the following proposition.

Proposition 4.1. *Let G be a countable group and T be a countable tree. Let $G \curvearrowright T$ be a (not necessarily strictly or uniformly) acylindrical action without a global fixed point and let $\mathcal{F} := \mathcal{F}_T \cup \mathcal{F}_\partial$ (as in Notation 3.5). Then the action $G \curvearrowright \Delta(T)$ of G on the Bowditch boundary of T is finitely \mathcal{F} -amenable.*

To make this paper more accessible to readers unfamiliar with the covering-construction-culture around the Farrell–Jones Conjecture, we postpone the proof of the above proposition to the Sections 6–10.

Proof of Theorem A (modulo the proof of Proposition 4.1). By Lemma 1.9, it suffices to prove the theorem for finitely generated G . Furthermore, if $G \curvearrowright T$ had a global fixed point, then $\mathcal{F} = \mathcal{ALL}$ and the conclusion of the theorem is imminent. Thus, we can assume without loss of generality that there is a minimal G -invariant subtree T_{\min} of T such that G acts cocompactly on T_{\min} . In particular, T_{\min} has only countably many edges and the restricted action is still acylindrical. If the Farrell–Jones Conjecture holds for a group G relative to a family \mathcal{G} , then it also holds for G relative to any larger family $\mathcal{G}' \supset \mathcal{G}$ (this is a special case of the transitivity principle [BFL14, Theorem 2.10]). Hence, it suffices to show the theorem for countable G and T and fixed point free acylindrical actions $G \curvearrowright T$.

Assuming Proposition 4.1, we proceed to show that the action of G on \overline{T}^{obs} is finitely \mathcal{F} -amenable: By Proposition 4.1, there is N' such that the action of G on the Bodwitch boundary $\Delta(T)$ is $N' - \mathcal{F}$ -amenable. Let $N := N' + 2$. To produce, for all $\alpha > 0$, an open G -invariant \mathcal{F} -cover of $G \times \overline{T}^{obs}$ that is wide and of dimension at most N , we first tend to covering $G \times \overline{T}^{obs} \setminus G \times \Delta(T)$.

For a moment we consider T with the path-metric topology and define two kinds of open sets of T . Let \mathcal{I} be all open edges of T and \mathcal{B} be all open balls of radius ϵ (i.e. some fixed small positive number) around vertices of T . Then, both \mathcal{I} and \mathcal{B} are G -invariant collections of open \mathcal{F} -subsets of T (with the path-metric topology). Since the observers' topology coincides with the path-metric topology on finite subtrees, it follows that $\mathcal{V}' := \{U \setminus V_\infty(T) \mid U \in \mathcal{I} \cup \mathcal{B}\}$ consists of open sets of \overline{T}^{obs} . Moreover, \mathcal{V}' is a G -invariant collection of open \mathcal{F} -subsets of \overline{T}^{obs} , since $V_\infty(T)$ is G -invariant. As $\mathcal{I} \cup \mathcal{B}$ is of order 1, so is \mathcal{V}' . Define $\mathcal{V} := \{G \times U' \mid U' \in \mathcal{V}'\}$. This is a G -invariant collection of open \mathcal{F} -subsets of $G \times \overline{T}^{obs}$ which still has order 1. For $\xi \in \overline{T}^{obs} \setminus \Delta(T)$ there is an $U' \in \mathcal{V}'$ with $\xi \in U'$. Thus—independent of $\alpha > 0$ —for all $(g, \xi) \in G \times (\overline{T}^{obs} \setminus \Delta(T))$ there is an element of \mathcal{V} that contains $B_\alpha(g) \times \{\xi\}$. In other words, \mathcal{V} is wide for $G \times (\overline{T}^{obs} \setminus \Delta(T))$.

Now, let $\alpha > 0$ be fixed. By Proposition 4.1, there is an open G -invariant \mathcal{F} -cover \mathcal{U}_α of $G \times \Delta(T)$ of dimension at most N' which is wide. Of course the sets in \mathcal{U}_α are not open sets of $G \times \overline{T}^{obs}$, but it is possible to thicken the collection \mathcal{U}_α to an open collection \mathcal{U}_α^+ of $G \times \overline{T}^{obs}$ without losing any of the desired properties (G -invariance, order, wideness) of \mathcal{U}_α (see [Bar17, Appendix B], [BL12b, Lemma 4.14]). Let \mathcal{U}_α^+ be the result of this thickening process. So \mathcal{U}_α^+ is a G -invariant collection of open \mathcal{F} -subsets of $G \times \overline{T}^{obs}$ that is wide for $G \times \Delta(T)$ and has order at most N' . Defining $\mathcal{V}_\alpha := \mathcal{V} \cup \mathcal{U}_\alpha^+$ gives the desired open G -invariant \mathcal{F} -cover of $G \times \overline{T}^{obs}$ which is wide for all of $G \times \overline{T}^{obs}$ and is of dimension at most $N = N' + 2$. Since we have now established finitely \mathcal{F} -amenability for the action $G \curvearrowright \overline{T}^{obs}$, to apply Theorem 1.8 we merely have to recall that \overline{T}^{obs} is a compact contractible controlled 1-dominated metric space (see Lemma 2.3, Lemma 2.4 and Lemma 2.6). \square

Inheritance properties now allow us to deduce the Farrell–Jones Conjecture relative to \mathcal{VCYC} if it is known (relative to \mathcal{VCYC}) for all groups in \mathcal{F} .

Corollary 4.2. *Let G be a group acting acylindrically on a simplicial tree T . If all vertex stabilisers of $G \curvearrowright T$ satisfy the K -theoretic Farrell–Jones Conjecture relative to \mathcal{VCYC} , then G satisfies the K -theoretic Farrell–Jones Conjecture relative to \mathcal{VCYC} .*

Proof. By Theorem A, G satisfies the K -theoretic Farrell–Jones Conjecture relative to $\mathcal{F}_T \cup \mathcal{F}_\partial$. Hence, by Lemma 3.6, G satisfies the K -theoretic Farrell–Jones Conjecture relative to the larger family $\mathcal{F}_T \cup \mathcal{VCYC}$. Since the Farrell–Jones Conjecture relative to \mathcal{VCYC} is closed under taking subgroups (see [BFL14, Theorem 2.8]), if $H \in \mathcal{F}_T$, then H satisfies the K -theoretic Farrell–Jones Conjecture relative to \mathcal{VCYC} by assumption. If $H \in \mathcal{VCYC}$, H satisfies the K -theoretic Farrell–Jones Conjecture trivially. Thus, by the transitivity principle for the Farrell–Jones Conjecture (see [BFL14, Theorem 2.10]) the claim follows. \square

The L -theoretic case works analogously, but we have to take overgroups of index 2 of vertex stabilisers of $G \curvearrowright T$ into account.

Corollary 4.3. *Let G be a group acting acylindrically on a simplicial tree T . Assume that all (subgroups of G that are) overgroups of index at most 2 of vertex stabilisers of $G \curvearrowright T$ satisfy the L -theoretic Farrell–Jones Conjecture relative to \mathcal{VCYC} . Then G satisfies the L -theoretic Farrell–Jones Conjecture relative to \mathcal{VCYC} .*

These two corollaries have a counterpart for the Farrell–Jones Conjecture with finite wreath products: A group G is said to satisfy the *Farrell–Jones Conjecture with finite wreath products relative to a family \mathcal{F}* if for all finite groups F the wreath product $G \wr F$ satisfies the Farrell–Jones Conjecture relative to \mathcal{F} . Recalling that we proved Theorem A basically by showing that the group G involved is strongly transfer reducible over $\mathcal{F}_T \cup \mathcal{F}_\partial$ (cf. the proof of Theorem 1.8), we can obtain the following variant of the last two corollaries by using [BLRR14, Theorem 5.1] and the transitivity principle.

Corollary 4.4. *Let G be a group acting acylindrically on a simplicial tree T . Assume that all vertex stabilisers of $G \curvearrowright T$ satisfy the Farrell–Jones Conjecture with finite wreath products relative to \mathcal{VCYC} . Then G satisfies the Farrell–Jones Conjecture with finite wreath products relative to \mathcal{VCYC} .*

5. THE PROOF OF THEOREM B

Theorem B. *Let $G = A *_C B$ act acylindrically on the associated Bass–Serre tree T . Let \mathcal{L} be a set of representatives for the orbits of the action $G \curvearrowright \partial T \times \partial T \setminus \text{diag}$. Then for any ring R there are isomorphisms*

$$\begin{aligned} \widetilde{\text{Nil}}_{n-1}(RC; R[A - C], R[B - C]) \\ \cong \bigoplus_{L \in \mathcal{L}} \text{coker}(H_n^{G_L}(E_{\mathcal{FIN}} G_L; \mathbf{K}_R) \rightarrow H_n^{G_L}(\text{pt}; \mathbf{K}_R)) \end{aligned}$$

for $n \in \mathbb{Z}$.

We now need some knowledge on the objects appearing in the K -theoretic Farrell–Jones Conjecture. In general, for any family \mathcal{G} of subgroups of G , $E_{\mathcal{G}}(G)$ denotes the classifying space for the family \mathcal{G} . Any G -CW-complex X with $X^H \simeq \text{pt}$ for $H \in \mathcal{G}$ and $X^H = \emptyset$ for $H \notin \mathcal{G}$ is a model for $E_{\mathcal{G}}G$.

We next give a suitable model for $E_{\mathcal{F}}G$, where $\mathcal{F} = \mathcal{F}_T \cup \mathcal{F}_\partial$ is again defined as in Notation 3.5. In particular, we do not yet require G to be an amalgamated product

or the action of G on T to be acylindrically. The author expects the following construction to be well known, but could not find a reference in the literature.

Let \mathcal{L} be a set of representatives for the orbits of the action $G \curvearrowright \partial T \times \partial T \setminus \text{diag}$. We think of \mathcal{L} as a collection of bi-infinite geodesic rays L in T and we denote by G_L the stabiliser of the element in \mathcal{L} given by L . In particular, each G_L is an element of \mathcal{F}_∂ . Denote by $\text{cone}(L)$ the simplicial complex (and hence CW-complex) obtained by taking the simplicial join $p_L * L$. Both L and $\text{cone}(L)$ are G_L -CW-complexes, and thus $G \times_{G_L} L$ and $G \times_{G_L} \text{cone}(L)$ are G -CW-complexes. Each $G \times_{G_L} L$, when viewed as a CW-complex, is the disjoint union of lines. The complex $G \times_{G_L} L$ contains one line for each element in the set $\{gL \mid g \in G\}$. Similarly, $G \times_{G_L} \text{cone}(L)$ viewed as a CW-complex is the disjoint union of coned off lines and contains one coned off line for each element in the set $\{\text{cone}(gL) \mid g \in G\}$.

Since T is a G -CW-complex as well, we can form the following pushout of G -CW-complexes, where i is the canonical inclusion induced by the inclusion of each line into its cone and j_L is the G -map given by sending $(g, t) \in G \times_{G_L} L$ to $gt \in gL \subset T$.

$$(1) \quad \begin{array}{ccc} \coprod_{L \in \mathcal{L}} G \times_{G_L} L & \xrightarrow{\coprod_{L \in \mathcal{L}} j_L} & T \\ \downarrow i & & \downarrow \\ \coprod_{L \in \mathcal{L}} G \times_{G_L} \text{cone}(L) & \longrightarrow & Y \end{array}$$

Lemma 5.1. *The G -CW-complex Y given by the above pushout is a model for $E_{\mathcal{F}}G$.*

Proof. The G -CW-complex Y is obtained from T by equivariantly gluing coned off lines onto T along their ‘baselines’. Thus, the intersection of (the images of) different cones in Y lies in T . In particular, if $L \neq L'$, then the cone point p_L over L is distinct from the cone point $p_{L'}$ over L' . It follows that each p_L is fixed exactly by the elements in G_L . Moreover, since Y is a G -CW-complex, elements of G that do not lie in some G_L can only fix points in T . Let $H \in \mathcal{F} = \mathcal{F}_T \cup \mathcal{F}_\partial$. If $H \notin \mathcal{F}_\partial$, then H fixes only points in T , and T^H is non-empty and contractible.

If $H \in \mathcal{F}_\partial$ and H contains an element of positive translation length (with respect to $G \curvearrowright T$), then H can not fix a point in T . Furthermore, such a group H fixes exactly one cone point p_L for a unique line L and—since it does not fix any point in T —it does not fix any other point in $\text{cone}(L)$. Hence, in this case $Y^H = p_L$.

If $H \in \mathcal{F}_\partial$ and all $h \in H$ have translation length zero, then any line L for which $Hp_L = p_L$ holds must be fixed pointwise by H . Hence, for any such line $\text{cone}(L)$ must be fixed pointwise by H as well. There might be more than one such line, but Y^H is still contractible: Since different cones can only intersect in T , we can first contract Y^H to T^H by simultaneously retracting all cones $\text{cone}(L)$ in Y^H to their respective lines L , which all lie in T^H . Then we contract T^H to a point.

Conversely, if for some $H \leq G$ the set Y^H is non-empty, then (since Y is a G -CW-complex) H must fix at least one 0-cell of Y . Thus, H either fixes at least one vertex of T or is a subgroup of at least one of the G_L . Then, by definition, H lies in $\mathcal{F}_T \cup \mathcal{F}_\partial = \mathcal{F}$. \square

In the sequel we will also use the fact that for every ring R there is an equivariant homology theory $H_*^?(-; \mathbf{K}_R)$ such that for all groups G the G -homology

theory $H_*^G(-; \mathbf{K}_R)$ is the G -homology theory appearing in the formulation of the K -theoretic Farrell-Jones Conjecture with coefficients in the ring R (see [BEL08, Theorem 6.1]). In particular, we assume the reader is comfortable with the properties of an equivariant homology theory (see [Lüc02, Section 1] for a definition).

Proof of Theorem B. Davis, Quinn and Reich, in [DQR11, Lemma 3.1], identified the group $\widetilde{\text{Nil}}_{n-1}(RC; R[A-C], R[B-C])$ with the cokernel of the map

$$\text{asmb}_n(G, \mathcal{F}_T, R) : H_n^G(E_{\mathcal{F}_T}G; \mathbf{K}_R) \rightarrow H_n^G(\text{pt}; \mathbf{K}_R).$$

By Theorem A a), this cokernel is isomorphic to the cokernel of the map

$$H_n^G(T; \mathbf{K}_R) = H_n^G(E_{\mathcal{F}_T}G; \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{F}}G; \mathbf{K}_R),$$

which can be determined by exploiting the properties of $H_*^G(-; \mathbf{K}_R)$ and the explicit model for $E_{\mathcal{F}}G$ constructed above. Applying the G -homology theory $H_*^G(-; \mathbf{K}_R)$ (abbreviated to $H_*^G(-)$ in the following) to the pushout (1) yields a Mayer-Vietoris sequence

$$\dots \rightarrow \bigoplus_{L \in \mathcal{L}} H_n^G(G \times_{G_L} L) \rightarrow H_n^G(T) \oplus \bigoplus_{L \in \mathcal{L}} H_n^G(G \times_{G_L} \text{cone}(L)) \rightarrow H_n^G(E_{\mathcal{F}}G) \rightarrow \dots$$

The space $\text{cone}(L)$ is G_L -equivariantly contractible and L is a model for $E_{\mathcal{F}LN}G_L$. Thus, using the induction structure of H_*^G for the G -CW-complexes $G \times_{G_L} L = \text{ind}_{G_L}^G L$ and $G \times_{G_L} \text{cone}(L) = \text{ind}_{G_L}^G(\text{cone}(L))$, the above sequence becomes

$$(2) \quad \dots \rightarrow \bigoplus_{L \in \mathcal{L}} H_n^{G_L}(E_{\mathcal{F}LN}G_L) \xrightarrow{(f_1, f_2)} H_n^G(T) \oplus \bigoplus_{L \in \mathcal{L}} H_n^{G_L}(\text{pt}) \rightarrow H_n^G(E_{\mathcal{F}}G) \rightarrow \dots$$

Furthermore, the map f_2 is induced by the projections $\text{pr}_L : E_{\mathcal{F}LN}G_L \rightarrow \text{pt}$. In other words, $f_2 = \bigoplus_{L \in \mathcal{L}} \text{asmb}_n(G_L, \mathcal{F}LN, R)$. Since the relative assembly map

$$H_*^{G'}(E_{\mathcal{F}LN}H) \rightarrow H_*^{G'}(E_{\text{VCYC}}H)$$

is split injective for all groups G' and rings R [Bar03, Theorem 1.3] and all G_L are virtually cyclic by Lemma 3.6, the maps $\text{asmb}_n(G_L, \mathcal{F}LN, R)$ are split injective. Hence, f_2 is split injective as well and the long exact sequence (2) gives rise to a short exact sequence

$$0 \rightarrow \bigoplus_{L \in \mathcal{L}} \mathcal{H}_n^{G_L}(E_{\mathcal{F}LN}G_L) \xrightarrow{(f_1, f_2)} \mathcal{H}_n^G(T) \oplus \bigoplus_{L \in \mathcal{L}} \mathcal{H}_n^{G_L}(\text{pt}) \rightarrow \mathcal{H}_n^G(E_{\mathcal{F}}G) \rightarrow 0$$

for every $n \in \mathbb{Z}$. By basic yoga with short exact sequences one can obtain from any short exact sequence of abelian groups

$$0 \rightarrow V_0 \xrightarrow{(i, j)} V_2 \oplus V_1 \rightarrow V_3 \rightarrow 0$$

with j injective, a short exact sequence of the form $0 \rightarrow V_2 \rightarrow V_3 \rightarrow \text{coker}(i) \rightarrow 0$. Thus, we obtain for all $n \in \mathbb{Z}$ the short exact sequence

$$0 \rightarrow H_n^G(T; \mathbf{K}_R) \xrightarrow{i_*} H_n^G(E_{\mathcal{F}}G; \mathbf{K}_R) \rightarrow \text{coker} \bigoplus_{L \in \mathcal{L}} \text{asmb}_n(G_L, \mathcal{F}LN, R) \rightarrow 0$$

from which the claim follows. \square

Exploiting what is known about the relative assembly map, one can now collect the following vanishing results for Waldhausen Nil-groups.

Corollary 5.2. *Let $G = A *_C B$ act acylindrically on its Bass-Serre tree T and let R be a regular ring. Then $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ vanishes rationally. Furthermore,*

- a) *if the action $G \curvearrowright T$ is strictly acylindrical (e.g. when C is malnormal in A or B), then $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C]) = 0$;*
- b) *if $\mathbb{Q} \subset R$, then $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C]) = 0$.*

Proof. By [LS16, Theorem 0.3], the relative assembly map

$$H_n^{G_L}(E_{\mathcal{FIN}}G_L; \mathbf{K}_R) \rightarrow H_n^{G_L}(E_{\text{VCYC}}G_L; \mathbf{K}_R)$$

is rationally bijective. Since all G_L are virtually cyclic, $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ vanishes rationally by Theorem B.

a) If $G \curvearrowright T$ is strictly acylindrical, then any G_L is either trivial or \mathbb{Z} . Since R is assumed to be regular, the fundamental theorem of algebraic K -theory [Ros94, 5.3.30 Theorem] implies that all $H_n^{G_L}(E_{\mathcal{FIN}}G_L; \mathbf{K}_R) \rightarrow H_n^{G_L}(\text{pt}; \mathbf{K}_R)$ are isomorphisms.

b) By [KL05, Lemma 21.24], the relative assembly map for G_L is integrally an isomorphism. Since all G_L are virtually cyclic, the claim follows. \square

In light of Waldhausen's notion of regular coherent groups (see [Wal78b, Section 19]) a remark on Corollary 5.2 a) is in order.

Remark 5.3. For all regular coherent groups C and all regular rings R , the group $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ already vanishes integrally by [Wal78b, Theorem 11.2]. However, it is easy to find examples of amalgamated free products $G = A *_C B$ acting strictly acylindrically on their Bass-Serre tree, such that C is not regular coherent, because it contains torsion: By [Kap99, Theorem C] any torsion-free hyperbolic group A' (that is not virtually cyclic) contains a subgroup $H \cong F_2$ that is malnormal in A' . Then $C = \mathbb{Z}/2\mathbb{Z} \times F_2$ is malnormal in $A := D_\infty \times A'$, but is not regular coherent.

Remark 5.4. In the cases where $\widetilde{\text{Nil}}_n(RC; R[A - C], R[B - C])$ does not vanish by any of the above statements, the right hand side of the isomorphisms in Theorem B has been further identified: By Lemma 3.6 the groups G_L that contribute to the direct sum are virtually cyclic of type I. For any such finite-by- \mathbb{Z} group $V = H \rtimes_\alpha \mathbb{Z}$ it is known (see [DQR11, Lemma 3.1]) that the cokernel of the relative assembly map $H_n^V(E_{\mathcal{FIN}}V; \mathbf{K}_R) \rightarrow H_n^V(E_{\text{VCYC}}V; \mathbf{K}_R)$ is isomorphic to the direct sum of two Farrell Nil-groups associated to RH and α . (These type of Nil-groups were introduced by Farrell in his PhD thesis [Far71] and a modern definition encompassing all degrees can be found in [Gru08]). In the case present, where H is finite, these Farrell Nil-groups are known to be either trivial or infinitely generated: in lower degrees, this was shown independently by Grunewald [Gru07] and Ramos [Ram07]. Very recently, Lafont, Prassidis and Wang extended this result to all degrees [LPW16] and obtained a structure result for these groups (provided they have finite exponent).

6. THE TACTIC FOR SHOWING FINITELY \mathcal{F} -AMENABILITY

The rest of this paper is concerned with the proof of Proposition 4.1. Thus, from now on, G is a countable group acting k -acylindrically and without a global fixed

point on a countable tree T . Without loss of generality we can assume $k \geq 1$, which spares us some notational exceptions.

Showing \mathcal{F} -amenability of the action $G \curvearrowright \Delta(T)$ amounts to providing intricate ‘wide’ covers for $G \times \Delta(T)$. A group G acting 1-acylindrically on a tree is relatively hyperbolic, so in this case the desired covers are already given by Bartels’ methods in [Bar17] and our tactic is to adapt those methods for the case of a k -acylindrical action.

In [Bar17], first a notion of Θ -small angles ‘between edges’ is introduced. Its crucial feature is that it functions like a proper G -invariant metric on the set of edges $E(T)$ incident to a given vertex. Then, a geodesic is called Θ -small if the angle between any two incident edges lying on this geodesic is Θ -small. In particular, any Θ -small geodesic ending in a vertex can only be extended in finitely many directions without loosing the property of being Θ -small. Bartels uses the notion of Θ -small geodesics to divide $G \times \Delta(T)$ in two subsets. For trees, the first one, $G \times_{\Theta} \partial T$, consists of all pairs $(g, \xi) \in G \times \partial T$ such that, starting from a fixed vertex in T of finite valence, the point $g^{-1}\xi$ can be reached with a Θ -small geodesic ray. So this set is of the form $G \cdot (\{1\} \times \partial T')$ for a locally finite subtree $T' \subset T$. Furthermore, by definition of a relatively hyperbolic group, G acts cocompactly on $E(T)$ and each edge stabiliser is finite. In the proof of Bartels, these three facts play an essential role in showing that $G \times_{\Theta} \partial T$ admits suitable covers. The second set, the rest, is covered by specially tailored sets that are constructed ad hoc, using the same notion of Θ -small angles.

In our case of groups acting k -acylindrically on trees, edge stabilisers are in general not finite. The entities that have finite stabilisers instead are the geodesic segments of length k in T . In trying to define a notion of Θ -small geodesics by measuring ‘angles’ between ‘incident segments of length k ’ on a geodesic, several difficulties arise. First, from a naïve geometric point of view, it is not clear when two segments of length k should be called incident. The second problem appears when determining which points $g^{-1}\xi \in \partial T$ can be reached from a given fixed vertex v_0 via a small geodesic. Since on a geodesic of length $< k$ there is no segment of length k , and thus no ‘angle’, any such geodesic is automatically Θ -small. Hence, one can start from v_0 in infinitely many directions along a Θ -small geodesic, and the set $G \times_{\Theta} \partial T$ would be of the form $G \cdot (\{1\} \times \partial T')$ where T' is a tree that is *not* locally finite. Moreover, the action of G on the set $E^k(T)$ of geodesic segments of length k is not cocompact anymore. We resolve these difficulties as follows.

In Section 7, a proper G -invariant metric on $E^k(T)$ is defined. This metric is constructed to have the additional property that the action of G on $E^k(T)$ is ‘cocompact along Θ -small geodesics’, and this property is sufficient for our purposes. Using this metric we can measure the distance between any two segments of length k . Our notion of Θ -small geodesics is also defined in Section 7, where ‘incident segments of length k ’ on a geodesic are segments on this geodesic whose midpoints have distance 1 in T . For the set $G \times_{\Theta} \partial T$, instead of starting to measure from a single vertex, we start to measure from one of two fixed geodesic segments of length k . These segments are ensured to be connected by a Θ -small geodesic. This way $G \times_{\Theta} \partial T$ can again be written as $G \cdot (\{1\} \times \partial T')$ for a locally finite subtree T' of T , and wide covers for $G \times_{\Theta} \partial T$ are constructed in Sections 9–10 utilising the long thin covers for coarse flow spaces provided by [Bar17, Theorem 1.1].

Covers for the remaining set $G \times \Delta(T) \setminus G \times_{\Theta} \partial T$ are defined ad hoc in Section 8. In general, a single such cover will not be wide, but we show that one can combine $k + 2$ of these covers to obtain a wide cover for $G \times \Delta(T) \setminus G \times_{\Theta} \partial T$. Since ‘incident segments of length k ’ overlap, this is more involved than in the case $k = 1$, and we elaborate on the difficulties that arise from overlapping segments at the beginning of Section 8.

Finally, formally sewing together the results of Sections 7-10 gives Proposition 4.1:

Proof of Proposition 4.1. The proposition follows immediately from Proposition 8.1 and Proposition 10.5 as for any given $\alpha > 0$ we can take the union of the cover \mathcal{W}_{α} from Proposition 8.1 and the cover $\mathcal{U}_{\Theta_{-1}(\alpha), \alpha}$ from Proposition 10.5. \square

7. DEFINING Θ -SMALL GEODESICS

In order to define Θ -small geodesics we need a G -invariant proper metric on the set $E^k(T)$ of all geodesic segments of length k in T whose endpoints are vertices of T . Since the action of G on $E^k(T)$ is, in general, not cocompact, we have to construct the metric in a way that the action on $E^k(T)$ becomes ‘cocompact along small geodesics’.

Lemma 7.1. *Let G be a countable group and X be a countable discrete space. Furthermore, let $G \curvearrowright X$ have finite point stabilisers. Let $\pi : X \rightarrow G \backslash X$ be the canonical projection and let $F = \{x_0, x_1, \dots\}$ be a set of representatives for the orbits of the action $G \curvearrowright X$. Then there is a G -invariant proper metric d_X on X compatible with the discrete topology such that the following holds: For all $\Theta > 0$ there is $n_0 \in \mathbb{N}$ such that if $x, y \in X$ with $d_X(x, y) \leq \Theta$, then $\pi(x) = [x_j], \pi(y) = [x_{j'}]$ for some $j, j' \leq n_0$. In (other) words: If x and y belong to different orbits and their distance is $\leq \Theta$, then the orbits of x and y both are among the first $n_0 = n_0(\Theta)$ orbits.*

Proof. Abels, Manoussos and Noskov showed that for a locally compact group H acting properly on a locally compact, σ -compact metrisable space Z , there is an H -invariant proper metric on Z compatible with the topology on Z [AMN11, Theorem 1.1]. Thus, in the situation of the lemma, the existence of a G -invariant proper metric on X follows directly from [AMN11, Theorem 1.1]. Their construction is split in several steps and one of those steps [AMN11, Section 7] assures that the constructed metric is orbitwise proper². This is obtained by taking any proper continuous function $f : G \backslash X \rightarrow [0, \infty)$ and adding $d'(x, y) := |f(\pi(x)) - f(\pi(y))|$ to the result of their previous construction step. Since in our case $G \backslash X$ is indeed countable (instead of only σ -countable), we can choose f such that orbits are forced further and further apart to obtain the additionally desired property the lemma claims. The rest of the construction of Abels, Manoussos and Noskov retains this property. A detailed elaboration on this modification can be found in the author’s PhD thesis [Kno16, Lemma 2.3]. \square

Recall that we assume G to act k -acylindrically on T for some $k \geq 1$, that the term *geodesic* refers to both finite geodesics as well as (bi-infinite) geodesic rays in

²For discrete spaces this amounts to saying that for all $r > 0$ and $x \in X$ the image of $B_r^d(x)$, the (closed) ball of radius r around x , under $\pi : X \rightarrow G \backslash X$ is finite.

T and that d_T denotes the path-metric on T . We set up the following long list of carefully crafted choices and notations, some of which are illustrated in Figure 2.

| | |
|---------------------------|---|
| $E^k(T)$ | Denote by $E^k(T)$ the set of all geodesic segments of length k in T whose endpoints are vertices. |
| $E^k(T) \cap S$ | For any subtree $S \subset T$ we write $\sigma \in E^k(T) \cap S$ if $\sigma \subset S$ and $\sigma \in E^k(T)$. |
| d_{E^k} | The set $E^k(T)$ is countable and the obvious action of G on $E^k(T)$ has finite point stabilisers, since the action $G \curvearrowright T$ is k -acylindrical. We fix a metric d_{E^k} on $E^k(T)$ according to Lemma 7.1. |
| σ lies on γ | We will often think of an element σ in $E^k(T)$ as an ‘elongated point’, which may or may not lie on a given geodesic γ . Thus, we will frequently use phrases like σ <i>lies on</i> γ , meaning that σ is a subset of the image of γ . |
| $o(\gamma), t(\gamma)$ | For a finite geodesic $\gamma : [a, b] \rightarrow T$ we denote by $o(\gamma) := \gamma(a)$ and $t(\gamma) := \gamma(b)$ its start- and endpoint, respectively. We also extend this notation to (bi-infinite) geodesic rays in the canonical way. |
| $m(\gamma), m(\sigma)$ | For a finite geodesic $\gamma : [a, b] \rightarrow T$ we denote by $m(\gamma) := \gamma(\frac{a+b}{2})$ its midpoint. Analogously, for an element $\sigma \in E^k(T)$ we denote by $m(\sigma)$ its midpoint. Note that, if k is even, the midpoint of σ is a vertex of T . |
| $\sigma_v(\gamma)$ | For a geodesic γ and any vertex v on γ we denote (if it exists) by $\sigma_v(\gamma)$ the geodesic segment of length k on $[o(\gamma), v]$ that ends in v . |
| $v+i, v-i$ | For a geodesic γ , any $i \in \mathbb{N}$ and a vertex v on γ we denote (if it exists) by $v+i$ (resp. $v-i$) the unique vertex on $[v, t(\gamma)]$ (resp. $[o(\gamma), v]$) that has distance i to v (with respect to d_T). If we want to specify along which geodesic we take this i th successor and i th predecessor, we write $v +_\gamma i$ and $v -_\gamma i$, respectively. |
| w_0, w'_0 | We fix two vertices $w_0, w'_0 \in V_0(T)$ with $d_T(w_0, w'_0) \geq 5k$. (Such points exist since we can always assume $V_0(T)$ to be unbounded with respect to d_T by replacing T with its barycentric subdivision.) |
| θ_0 | Let $\theta_0 := \max\{d_{E^k}(\sigma, \sigma') \mid \sigma, \sigma' \in E^k(T) \cap [w_0, w'_0]\}$. |
| m_ϵ | Choose your favourite $\epsilon < \frac{1}{2}$. For the next two lines denote by m the midpoint of $[w_0, w'_0]$. We denote (throughout the rest of this paper) by m_ϵ the unique point on $[w_0, m]$ that has distance ϵ to m . |
| $a(g, \xi)$ | Now for every $(g, \xi) \in G \times \Delta(T)$ only one of the two geodesics $[gw_0, \xi]$ and $[gw'_0, \xi]$ contains the point gm_ϵ . We denote by $a(g, \xi) \in \{gw_0, gw'_0\}$ the vertex such that $[a(g, \xi), \xi]$ contains gm_ϵ . Note that, for all $g, h \in G$, we have $ha(g, \xi) = a(hg, h\xi)$. |

Next we use the metric d_{E^k} to define when a geodesic is Θ -small.

Definition 7.2. Let $\Theta > 0$ and let γ be a geodesic in T . Any term of the form $d_{E^k}(\sigma, \sigma')$ for $\sigma, \sigma' \in E^k(T) \cap \text{im}(\gamma)$ with $\text{diam}_{d_T}(\sigma \cap \sigma') = k - 1$ is a *measurement (on γ)*. For a vertex v on γ the term $d_{E^k}(\sigma_v(\gamma), \sigma_{v+1}(\gamma))$ is called the *measurement at v (on γ)*. This is illustrated in Figure 3.

A measurement is Θ -small if it is $\leq \Theta$ and it is Θ -large if it is $> \Theta$. Finally, the geodesic γ is Θ -small if all measurements on γ are Θ -small. In particular, any geodesic γ of length $\leq k$ is automatically Θ -small, since in this case there are no measurements on γ .

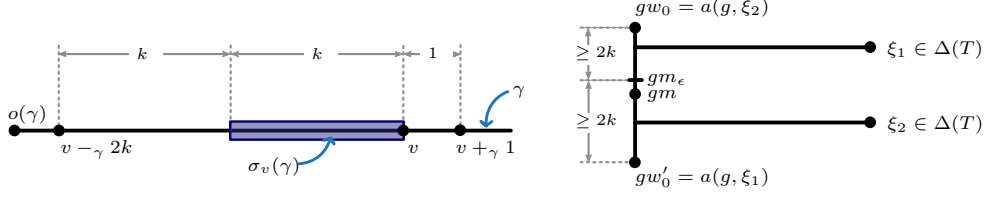


FIGURE 2. The vertices $v - \gamma 2k$ and $v + \gamma 1$ and the segment $\sigma_v(\gamma)$ on γ . The vertex $a(g, \xi)$ is well-defined, since the point gm_ϵ is not a vertex.

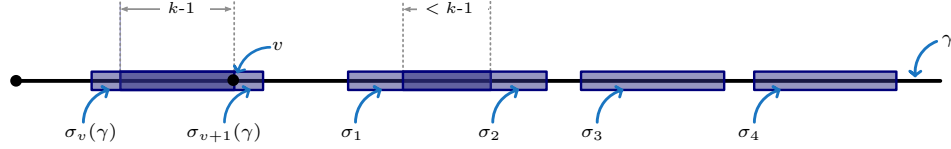


FIGURE 3. $d_{E^k}(\sigma_v(\gamma), \sigma_{v+1}(\gamma))$ is the measurement at v on γ . Neither $d_{E^k}(\sigma_1, \sigma_2)$ nor $d_{E^k}(\sigma_3, \sigma_4)$ are measurements.

Notation 7.3. For $\Theta > 0$ we define

$$G \times_\Theta \partial T := \{(g, \xi) \in G \times \partial T \mid [a(g, \xi), \xi] \text{ is } \Theta\text{-small}\}.$$

We will find covers for $G \times_\Theta \partial T$ and its complement in $G \times \Delta(T)$ separately.

8. COVERS FOR $G \times \Delta(T) \setminus G \times_\Theta \partial T$

The goal of this section is to establish the following proposition.

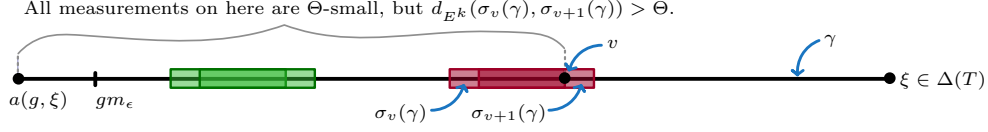
Proposition 8.1. *Retain the assumptions of Proposition 4.1. Then there is $N \in \mathbb{N}$ such that for all $\alpha > 0$ there is a $\Theta_{-1} = \Theta_{-1}(\alpha) \in \mathbb{R}$ and a G -invariant collection \mathcal{W}_α of open \mathcal{F}_T -subsets of $G \times \Delta(T)$ such that the order of \mathcal{W}_α is at most N and \mathcal{W}_α is wide for $G \times \Delta(T) \setminus G \times_{\Theta_{-1}(\alpha)} \partial T$.*

We start by defining ad hoc the sets that will later on be used to build the collection \mathcal{W}_α .

Definition 8.2. Let $\Theta \geq \theta_0$ and $v \in V(T)$. Define the set $W(v, \Theta) \subset G \times \Delta(T)$ to be

$$\begin{aligned} W(v, \Theta) := \{ (g, \xi) \in G \times \Delta(T) \mid & v \in [gm_\epsilon, \xi], [a(g, \xi), v] \text{ is } \Theta\text{-small}, \\ & v = \xi \text{ or } d_{E^k}(\sigma_v(\gamma), \sigma_{v+\gamma 1}(\gamma)) > \Theta, \\ & \text{where } \gamma := [a(g, \xi), \xi] \}. \end{aligned}$$

This definition is illustrated below in Figure 4. Furthermore, note that, if $v \in [gm_\epsilon, \xi]$, then the geodesic $[a(g, \xi), v]$ is of length $\geq 2k$ since we have chosen w_0, w'_0 with $d_T(w_0, w'_0) \geq 5k$. Hence, if $v \in [gm_\epsilon, \xi]$, then the segment $\sigma_v(\gamma)$ is always defined and, if additionally $v \neq \xi$, so is the segment $\sigma_{v+\gamma 1}(\gamma)$.


 FIGURE 4. The point (g, ξ) lies in $W(v, \Theta)$.

If we were to drop the requirement of wideness in Proposition 8.1, then one collection of the form $\mathcal{W}_\Theta := \{W(v, \Theta) \mid v \in V(T)\}$ would be sufficient as the following two lemmata show.

Lemma 8.3. *Let $\Theta \geq \theta_0$.*

- a) *For all $v \in V(T)$, the set $W(v, \Theta) \subset G \times \Delta(T)$ is open.*
- b) *For all $g \in G$ and $v \in V(T)$, we have $gW(v, \Theta) = W(gv, \Theta)$.*
- c) *If $v \neq w \in V(T)$, then $W(v, \Theta) \cap W(w, \Theta) = \emptyset$.*

Proof. a) Let $(g, \xi) \in W(v, \Theta)$ and let $\gamma := [a(g, \xi), \xi]$. First, let $v \neq \xi$, and thus $[a(g, \xi), v +_\gamma 1] \subset [a(g, \xi), \xi]$. Note that in this case any point (g, ξ') with

$$[a(g, \xi), v +_\gamma 1] = [a(g, \xi'), v +_\gamma 1] \subseteq [a(g, \xi'), \xi']$$

lies in $W(v, \Theta)$ as well (since the measurements relevant for $(g, \xi) \in W(v, \Theta)$ are the same measurements relevant for $(g, \xi') \in W(v, \Theta)$). Set $U := M(\xi, v) \cap \Delta(T)$. Then, for all $\xi' \in U$, the geodesic $[a(g, \xi), \xi']$ contains $v +_\gamma 1$, and hence gm_ϵ as well. Therefore, $[a(g, \xi), v +_\gamma 1] = [a(g, \xi'), v +_\gamma 1]$ and $(g, \xi) \in W(v, \Theta)$ follows. If $v = \xi$, then we construct U as follows: Let $\sigma := \sigma_v([a(g, \xi), v])$. Since d_{E^k} is proper, there are only finitely many σ' with $d_{E^k}(\sigma, \sigma') \leq \Theta$ and $\text{diam}_{d_T}(\sigma \cap \sigma') = k - 1$. Denote them by $\sigma'_1, \dots, \sigma'_m$. Set

$$U := M(v, \{t(\sigma'_1), \dots, t(\sigma'_m)\}) \cap \Delta(T).$$

Since $[a(g, \xi), v]$ is Θ -small by assumption, the segment $\sigma_{v-\gamma 1}(\gamma)$ is some σ'_i , and hence $v -_\gamma 1$ is $t(\sigma'_i)$. Therefore, for any $\xi' \in U$, the geodesic $[a(g, \xi), \xi']$ contains $[a(g, \xi), v]$ and, in particular, gm_ϵ . It follows that $[a(g, \xi'), v] = [a(g, \xi), v]$, which is Θ -small by assumption. And our choice of the σ'_i guarantees that the measurement at v on $[a(g, \xi'), \xi']$ is Θ -large.

b) This follows from the G -invariance of d_{E^k} .

c) Assume we have $v \neq w$ and $(g, \xi) \in W(v, \Theta) \cap W(w, \Theta)$. Then, by definition of $W(v, \Theta)$ and $W(w, \Theta)$, both v and w lie on $[gm_\epsilon, \xi]$. Hence, without loss of generality we may assume v lies on $[gm_\epsilon, w]$. Now, on the one hand, $(g, \xi) \in W(w, \Theta)$ implies that the measurement at v on $[a(g, \xi), w]$ is Θ -small. On the other hand, $(g, \xi) \in W(v, \Theta)$ implies that the measurement at v on $[a(g, \xi), \xi]$ is Θ -large. \square

Lemma 8.4. *Let $\Theta \geq \theta_0$. Then $\mathcal{W}_\Theta := \{W(v, \Theta) \mid v \in V(T)\}$ is a G -invariant collection of open \mathcal{F}_T -subsets of $G \times \Delta(T)$ and covers $G \times \Delta(T) \setminus G \times_\Theta \partial T$. Furthermore, the order of \mathcal{W}_Θ is 0.*

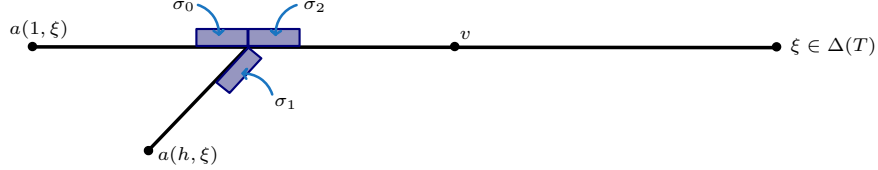
Proof. By Lemma 8.3 it is clear that \mathcal{W}_Θ is an open, G -invariant collection of order 0. Moreover, by combining Lemma 8.3 b) and c), it follows that each $W(v, \Theta)$ is an \mathcal{F}_T -subset of $G \times \Delta(T)$. The only property left to show is that the collection \mathcal{W}_Θ covers the set $G \times \Delta(T) \setminus G \times_\Theta \partial T$: Let $(g, \xi) \in G \times \Delta(T) \setminus G \times_\Theta \partial T$ and $\gamma := [a(g, \xi), \xi]$. If γ is Θ -small, then $\xi \in V(T)$, and hence $(g, \xi) \in W(\xi, \Theta)$. If γ is

not Θ -small, then there is a unique vertex v on γ such that $[a(g, \xi), v]$ is Θ -small, but $[a(g, \xi), v +_\gamma 1]$ is not. Note that v might lie on $[gw_0, gw'_0]$, but due to our choice of θ_0 the vertex v can not lie between $a(g, \xi)$ and gm_ϵ . Thus, $v \in [gm_\epsilon, \xi]$ and $(g, \xi) \in W(v, \Theta)$ holds. \square

To obtain—for a given $\alpha > 0$ —a threshold $\Theta_{-1}(\alpha)$ and a collection \mathcal{W}_α that is wide for $G \times \Delta(T) \setminus G \times_{\Theta_{-1}(\alpha)} \partial T$, it is necessary to combine $k + 2$ collections of the form \mathcal{W}_{Θ_i} . The verification that we can indeed find suitable Θ_i such that $\mathcal{W}_\alpha := \bigcup_{i=0}^{k+1} \mathcal{W}_{\Theta_i}$ is wide for $G \times \Delta(T) \setminus G \times_\Theta \partial T$ is rather technical, so we outline its underlying ideas first: Since all \mathcal{W}_Θ are G -invariant and all $G \times \Delta(T) \setminus G \times_\Theta \partial T$ are as well, it suffices to consider wideness for points of the form $(1, \xi)$.

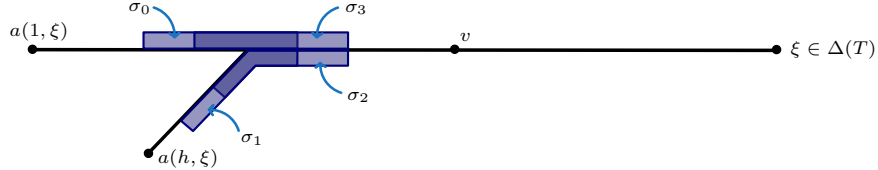
For any $\Theta \geq \theta_0$ we already know that for each point $(1, \xi) \in G \times \Delta(T) \setminus G \times_\Theta \partial T$ we can find a vertex $v \in V(T)$ by Lemma 8.4 such that $(1, \xi) \in W(v, \Theta)$ holds. We do not expect $[a(h, \xi), v]$ to be Θ -small for $h \in B_\alpha(1)$ as well, but we wish to calculate how small the geodesics $[a(h, \xi), v]$ are, given that $[a(1, \xi), v]$ is Θ -small.

In the case $k = 1$ this results in the following type of picture:



Let L_0 be the tree spanned by $\{a(h, \xi) \mid h \in B_\alpha(1)\}$. Since $B_\alpha(1)$ is finite, L_0 is as well. Thus, there is some Υ independent of h such that $d_{E^1}(\sigma_0, \sigma_1) \leq \Upsilon$. Since $d_{E^1}(\sigma_0, \sigma_2) \leq \Theta$ by virtue of $(1, \xi) \in W(v, \Theta)$, we can estimate $d_{E^1}(\sigma_1, \sigma_2)$ to be $\leq \Theta + \Upsilon$. Hence, for all $h \in B_\alpha(1)$ the geodesic $[a(h, \xi), v]$ is $\Theta + \Upsilon$ -small. If the measurement at v on $[a(1, \xi), \xi]$ is not only Θ -large, but in fact $\Theta + \Upsilon$ -large, we can conclude $B_\alpha(1) \times \{\xi\} \subset W(v, \Theta + \Upsilon)$. However, in general, the measurement at v on $[a(1, \xi), \xi]$ is not $\Theta + \Upsilon$ -large and this requires that we ‘push v further to the right’. More precisely, we search for the first vertex w on $[a(1, \xi), \xi]$ such that $[a(1, \xi), w]$ is $\Theta + \Upsilon$ -small, but the measurement at w is not. If such w exists, then $B_\alpha(1) \times \{\xi\} \subset W(w, \Theta + \Upsilon)$ follows. If such w does not exist, then $[a(1, \xi), \xi]$ is $\Theta + \Upsilon$ -small. If $\xi \in V_\infty(T)$, our tailored definition of the sets $W(v, \Theta + \Upsilon)$ implies that $B_\alpha(1) \times \{\xi\} \subseteq W(\xi, \Theta + \Upsilon)$ holds. And if $\xi \in \partial T$, we do not require our collection of sets to be wide at $(1, \xi)$ anyway. Since we only used sets of the form $W(-, \Theta + \Upsilon)$, we thus obtain a wide collection for $G \times \Delta(T) \setminus G \times_\Theta \partial T$ and this basically concludes the proof of Proposition 8.1 for the case $k = 1$.

In the case $k \geq 2$, however, examining how small the geodesics $[a(h, \xi), v]$ are is more involved since we will have to make statements about pictures of the following type:



We still know that L_0 (as defined before) is finite. Thus, we still have a bound Υ on all measurements on $[a(1, \xi), a(h, \xi)]$. However, we do not have a triangle inequality anymore to give an estimate for $d_{E^k}(\sigma_1, \sigma_2)$ based on Υ and the fact

that $d_{E^k}(\sigma_0, \sigma_3) \leq \Theta$. So in order to obtain some estimate we use that d_{E^k} is proper: since $[a(1, \xi), v]$ is Θ -small, has a segment of length $\geq k$ in common with L_0 and the latter is finite, there are only finitely many possibilities for σ_3 . Adding all possible σ_3 to L_0 still results in a finite tree L_1 and we find some bound Υ' on the measurements lying on L_1 . This obviates the need for a triangle inequality, since now both σ_1 and σ_2 lie on L_1 . Hence, $d_{E^k}(\sigma_1, \sigma_2)$ is Υ' -small. The idea of ‘pushing v further to the right if necessary’ stays the same, but various possible constellations of how the resulting vertex w lies relative to v necessitates more than one iteration step and careful case distinctions, both of which will be carried out in the following.

We start by making precise how to define an enlargement of L_0 suitable for all iteration steps.

Choosing Constants 8.5. Let $\alpha > 0$. The following notions all depend on α . However, since we will never compare different values of α , this dependence is not reflected in the notation.

| | |
|--------------------|--|
| L_0 | The set $B_\alpha(1)$ is finite. As is the set $M := \{hw_0, hw'_0 \mid h \in B_\alpha(1)\}$. Let $L_0 \subset T$ be the subtree of T spanned by M . In particular, L_0 is a finite subtree of T . |
| Υ_0 | We define Υ_0 as the diameter of the set $E^k(T) \cap L_0$ with respect to the metric d_{E^k} . Thus, for all $\sigma, \sigma' \in E^k(T) \cap L_0$ the inequality $d_{E^k}(\sigma, \sigma') \leq \Upsilon_0$ holds. Note that Υ_0 automatically has to be $\geq \theta_0$ as $[w_0, w'_0] \subset L_0$. |
| $L_k(\chi)$ | For $\chi > 0$ we construct the complex $L_k(\chi)$ by enlarging $L_0 =: L_0(\chi)$ iteratively k -times as follows: For $1 \leq \nu \leq k$ let $L_\nu(\chi)$ be the union of $L_{\nu-1}(\chi)$ and all $\sigma \in E^k(T)$ such that there is $\sigma' \in E^k(T) \cap L_{\nu-1}(\chi)$ with $\sigma \cap \sigma' \neq \emptyset$ and $d_{E^k}(\sigma, \sigma') \leq \chi$. Since, to obtain $L_\nu(\chi)$ from $L_{\nu-1}(\chi)$, we only add segments that have non-trivial intersection with $L_{\nu-1}(\chi)$, each $L_\nu(\chi)$ is again a subtree of T . Since L_0 is finite and d_{E^k} is proper, the tree $L_\nu(\chi)$ is finite for all $1 \leq \nu \leq k$. |
| $\Upsilon_1(\chi)$ | We define $\Upsilon_1(\chi)$ as the diameter of the set $E^k(T) \cap L_k(\chi)$ with respect to the metric d_{E^k} . Thus, for all $\sigma, \sigma' \in E^k(T) \cap L_k(\chi)$ the inequality $d_{E^k}(\sigma, \sigma') \leq \Upsilon_1(\chi)$ holds. Note that $\Upsilon_1(\chi) \geq \Upsilon_0$. |
| $\Upsilon_m(\chi)$ | Let $\chi > 0$. For $m \geq 0$ we define $\Upsilon_m(\chi)$ iteratively by $\Upsilon_m(\chi) := \Upsilon_1(\Upsilon_{m-1}(\chi))$. |

The proof of Proposition 8.1 requires some more notation.

Notation 8.6. For a given $\alpha > 0$ and $(1, \xi) \in G \times \Delta(T)$ we fix the following notation that depends on α and ξ (cf. Figure 5 for an illustration of some of the notation). The naming will not always reflect both of these dependencies, though.

| | |
|--------------------|--|
| γ_h | For $h \in B_\alpha(1)$ let $\gamma_h := [a(h, \xi), \xi]$. |
| v_h | For $h \neq 1$ let v_h be the unique first vertex that lies on both γ_1 and γ_h . Hence, we have $\gamma_1 \cap \gamma_h = [v_h, \xi]$. Note that the case $v_h = \xi$ is possible. |
| B_v^{inn} | For any vertex v of γ_1 we set $B_v^{\text{inn}} := \{h \in B_\alpha(1) \mid v \in V(\gamma_h)\}$. Thus, $h \in B_\alpha(1)$ is an element of B_v^{inn} if and only if γ_h contains v , which is the case if and only if $[v, \xi] \subseteq [v_h, \xi]$. |
| B_v^{ext} | Furthermore, let $B_v^{\text{ext}} := B_\alpha(1) \setminus B_v^{\text{inn}}$. Thus, $h \in B_\alpha(1)$ is an element of B_v^{ext} if and only if $v \notin [v_h, \xi]$. |

$v_0(\chi)$ For any $\chi > \theta_0$ denote—if it exists—by $v_0(\chi)$ the first vertex on γ_1 such that $[a(1, \xi), v_0(\chi)]$ is χ -small, but $[a(1, \xi), v_0(\chi) +_{\gamma_1} 1]$ is not. In particular, the existence of $v_0(\chi) +_{\gamma_1} 1$ is necessary for the existence of $v_0(\chi)$. Note that $d_T(a(1, \xi), v_0(\chi)) \geq 2k$: Since $[w_0, w'_0]$ is θ_0 -small, it follows that $m_\epsilon \in [a(1, \xi), v_0(\chi)]$ and $[a(1, \xi), m_\epsilon]$ has length $\geq 2k$.

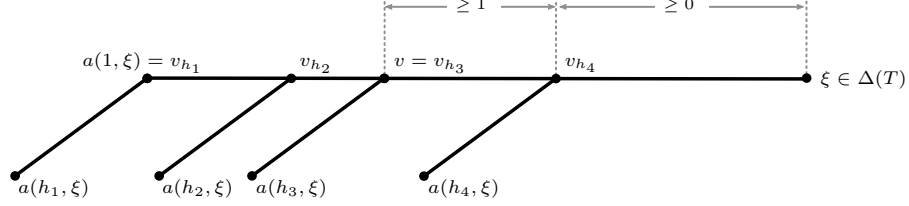


FIGURE 5. $h_1, h_2, h_3 \in B_\alpha(1)$ belong to B_v^{inn} , but $h_4 \in B_\alpha(1)$ belongs to B_v^{ext} .

We proceed by showing a series of technical lemmata, whose sole purpose is to increase readability of the proof of Proposition 8.1. For all statements we will use extensively the constants defined in Choosing Constants 8.5 and assume that α and $(1, \xi)$ are fixed. While looking at the pictures in the proofs it is also important to keep in mind the following observation: Our definition of $W(v, \Theta)$ allows for ξ to lie on $[w_0, w'_0]$. However, in what follows we often assume the existence of a $v_0(\chi)$ for some $\chi \geq \Upsilon_0$, which forces ξ to lie outside of $[w_0, w'_0]$.

Lemma 8.7. *Let $\chi \geq \Upsilon_0$. If $v_0(\chi)$ exists, then $B_{v_0(\chi)}^{\text{inn}} = B_\alpha(1)$.*

Proof. Assume there is $h \in B_\alpha(1)$ such that $h \notin B_{v_0(\chi)}^{\text{inn}}$. Thus, $v_h \in (v_0(\chi), \xi]$. Then by uniqueness of geodesics in T and the construction of Υ_0 it follows that $[a(1, \xi), v_0(\chi) +_{\gamma_1} 1]$ is Υ_0 -small (cf. Figure 5 with $h = h_4$ and $v_0(\chi) = v$). A contradiction. \square

In general, the ‘concatenation’ of two χ -small geodesics $[x, y]$ (ending in a vertex of T) and $[y, z]$ (starting in a vertex of T) is not χ -small. However, if two χ -small geodesics overlap in a sufficiently large terminal segment, we can conclude in this way. This is made precise in the following lemma, whose proof is entirely encapsulated in Figure 6.

Lemma 8.8. *Let $\chi \geq 0$ and let $[x, z]$ be a geodesic. If there are y, y' on $[x, z]$ such that the intersection $[x, y] \cap [y', z] = [y', y]$ has length $\geq k$ and both $[x, y]$ and $[y', z]$ are χ -small, then $[x, z]$ is χ -small.*

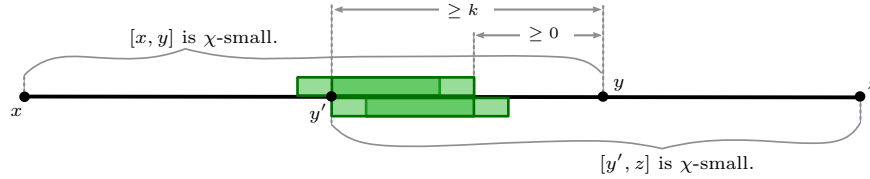


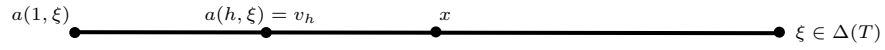
FIGURE 6. The ‘union’ of two χ -small geodesics overlapping in a segment of length $\geq k$ is again χ -small.

In the sequel we will use this way of reasoning without explicitly referencing the above lemma.

Lemma 8.9. *Let $\chi \geq \Upsilon_0$ and x be a vertex on γ_1 such that $[a(1, \xi), x]$ is χ -small. Then the geodesic $[a(h, \xi), x]$ is $\Upsilon_1(\chi)$ -small for all $h \in B_\alpha(1)$.*

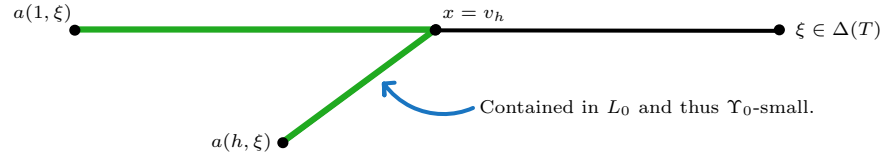
Proof. If $h \in B_x^{\text{ext}}$, then $[a(h, \xi), x] \subset [a(h, \xi), a(1, \xi)] \subset L_0$. Hence, $[a(h, \xi), x]$ is Υ_0 -small. If $h \in B_x^{\text{inn}}$, then we distinguish cases upon where v_h lies on $[a(1, \xi), x]$:

a) This case is clear:

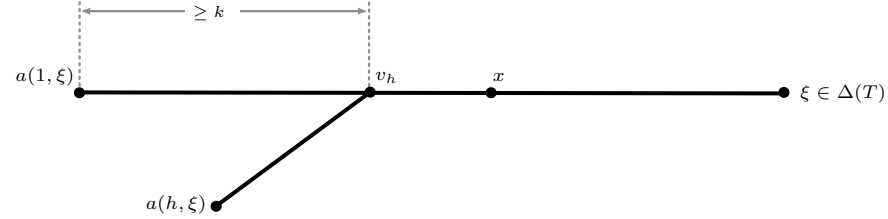


Thus, we assume from now on, that $a(h, \xi) \notin [a(1, \xi), x]$.

b) This one is clear as well:

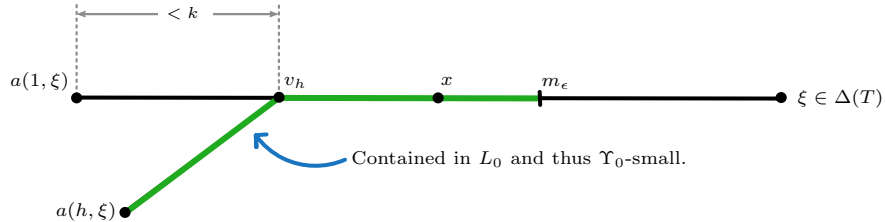


c) If v_h lies in the interior of $[a(1, \xi), x]$ and $[a(1, \xi), v_h]$ has length $\geq k$,

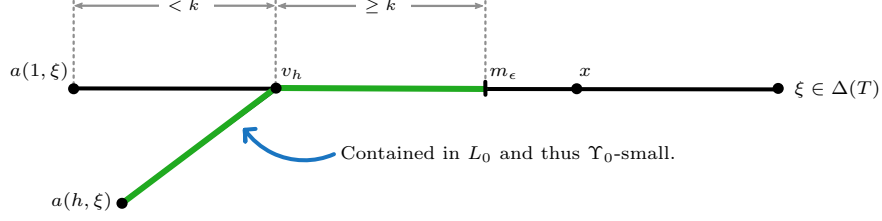


then the first k edges on $[v_h, x]$ (if there are this many) are added to L_0 in the process of building $L_k(\chi)$, since $[a(1, \xi), v_h]$ is χ -small (cf. Choosing Constants 8.5). Thus, $[a(h, \xi), v_h +_{\gamma_1} k]$ (resp. $[a(h, \xi), x]$ if $[v_h, x]$ is of length $< k$) is $\Upsilon_1(\chi)$ -small. Since $[v_h, x]$ was χ -small by assumption, it follows that $[a(h, \xi), x]$ is $\Upsilon_1(\chi)$ -small.

d) If v_h lies in the interior of $[a(1, \xi), x]$ and $[a(1, \xi), v_h]$ has length $< k$, there are two subcases. The subcase where $x \in [a(1, \xi), m_\epsilon]$ is trivial:

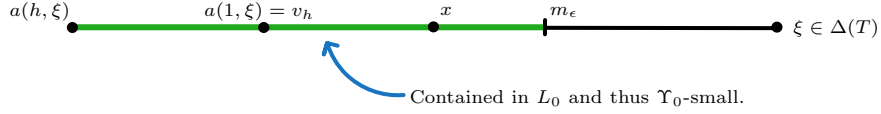


The subcase where $x \notin [a(1, \xi), m_\epsilon]$ looks as follows:

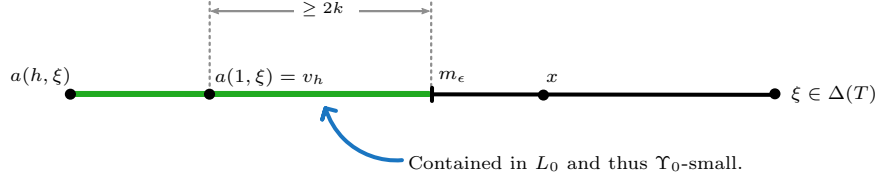


Since $[v_h, m_\epsilon]$ has length $\geq k$, $[a(h, \xi), m_\epsilon]$ is Υ_0 -small and $[v_h, x]$ is χ -small, the geodesic $[a(h, \xi), x]$ is χ -small.

e) This case is trivial again:



f) And finally:



Again, the geodesic $[a(h, \xi), x]$ is χ -small since $[a(h, \xi), m_\epsilon]$ is Υ_0 -small, the geodesic $[a(1, \xi), x]$ is χ -small and $[a(1, \xi), m_\epsilon]$ has length $\geq k$. \square

Lemma 8.10. *Let $\chi \geq \Upsilon_0$. If $v_0(\chi)$ exists and $B_{v_0(\chi)-\gamma_1 k}^{\text{inn}} = B_\alpha(1)$, then one of the following statements holds:*

- 1) $B_\alpha(1) \times \{\xi\} \subset W(v_0(\chi), \Upsilon_1(\chi))$;
- 2) $v_0(\Upsilon_1(\chi))$ does not exist, i.e. $[a(1, \xi), \xi]$ is $\Upsilon_1(\chi)$ -small;
- 3) $v_0(\Upsilon_1(\chi))$ exists and $B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_1(\chi)), \Upsilon_1(\chi))$.

Proof. In this proof we abbreviate d_{E^k} to d . For all $h \in B_\alpha(1)$, the geodesic $[a(h, \xi), v_0(\chi)]$ is $\Upsilon_1(\chi)$ -small by Lemma 8.9 (applied to $x = v_0(\chi)$). Since $B_\alpha(1) = B_{v_0(\chi)-\gamma_1 k}^{\text{inn}}$, the geodesics $[v_h, v_0(\chi)] \subset \gamma_h$ all have length $\geq k$. Hence, for all $h \in B_\alpha(1)$, the measurement at $v_0(\chi)$ on γ_1 is the measurement at $v_0(\chi)$ on γ_h . Thus, if the measurement at $v_0(\chi)$ on γ_1 is $\Upsilon_1(\chi)$ -large, so is the measurement at $v_0(\chi)$ on γ_h and $B_\alpha(1) \times \{\xi\} \subset W(v_0(\chi), \Upsilon_1(\chi))$ follows, which is statement 1) above.

Assume from now on that the measurement at $v_0(\chi)$ on γ_1 is $\Upsilon_1(\chi)$ -small. If $v_0(\Upsilon_1(\chi))$ does not exist, then $[a(1, \xi), \xi]$ is $\Upsilon_1(\chi)$ -small, which is statement 2) above. If $v_0(\Upsilon_1(\chi))$ exists, it lies in $(v_0(\chi), \xi]$. Thus, the intersection of the $\Upsilon_1(\chi)$ -small geodesic $[a(h, \xi), v_0(\chi)]$ with the $\Upsilon_1(\chi)$ -small geodesic $[v_0(\chi)-\gamma_1 k, v_0(\Upsilon_1(\chi))]$ has length $\geq k$ for all $h \in B_\alpha(1)$. Thus, all $[a(h, \xi), v_0(\Upsilon_1(\chi))]$ are still $\Upsilon_1(\chi)$ -small and the measurement at $v_0(\Upsilon_1(\chi))$ on γ_1 is the measurement at $v_0(\Upsilon_1(\chi))$ on γ_h . Hence, $B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_1(\chi)), \Upsilon_1(\chi))$, which is statement 3) above. \square

Lemma 8.11. *Let $\chi \geq \Upsilon_0$. Assume that $v_0(\chi)$ exists and let $0 \leq r \leq k-1$ such that $B_{v_0(\chi)-\gamma_1 r}^{\text{ext}} = \emptyset$, but $B_{v_0(\chi)-\gamma_1(r+1)}^{\text{ext}} \neq \emptyset$. Then one of the following statements holds:*

- 1) $B_\alpha(1) \times \{\xi\} \subset W(v_0(\chi), \Upsilon_1(\chi))$;
- 2) $v_0(\Upsilon_2(\chi))$ does not exist, i.e. $[a(1, \xi), \xi]$ is $\Upsilon_2(\chi)$ -small;
- 3) $v_0(\Upsilon_2(\chi))$ exists and there is an $s \geq 1$ such that $B_{v_0(\Upsilon_2(\chi))-\gamma_1(r+s)}^{\text{ext}} = \emptyset$, but $B_{v_0(\Upsilon_2(\chi))-\gamma_1(r+s+1)}^{\text{ext}} \neq \emptyset$.

Proof. In this proof we abbreviate d_{E^k} to d again. As in the previous lemma, for all $h \in B_\alpha(1)$ the geodesics $[a(h, \xi), v_0(\chi)]$ is $\Upsilon_1(\chi)$ -small by Lemma 8.9 (applied to $x = v_0(\chi)$). The measurement at $v_0(\chi)$ on $[a(h, \xi), \xi]$ is

$$d(\sigma_{v_0(\chi)}(\gamma_h), \sigma_{v_0(\chi)+1}(\gamma_h)).$$

Note that, since $B_{v_0(\chi)}^{\text{inn}} = B_\alpha(1)$ by Lemma 8.7, $v_0(\chi) + \gamma_1 1 = v_0(\chi) + \gamma_h 1$. So the $+1$ in the second argument above is unambiguous. However, in general $\sigma_{v_0(\chi)+1}(\gamma_h) \neq \sigma_{v_0(\chi)+1}(\gamma_1)$ and this is the point where we have to put in a little work.

If for all $h \in B_\alpha(1)$ the measurement at $v_0(\chi)$ on γ_h is $\Upsilon_1(\chi)$ -large, then

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\chi), \Upsilon_1(\chi))$$

by definition of $W(v_0(\chi), \Upsilon_1(\chi))$, and this is statement 1) above. So assume from now on that there is an $h_1 \in B_\alpha(1)$ such that the measurement at $v_0(\chi)$ on γ_{h_1} is $\Upsilon_1(\chi)$ -small. Moreover, we can assume that $v_0(\Upsilon_2(\chi))$ does exist, since the case that $v_0(\Upsilon_2(\chi))$ does not exist, is exactly statement 2) above.

Since $B_{v_0(\chi)-\gamma_1 r}^{\text{ext}} = \emptyset$, but $B_{v_0(\chi)-\gamma_1(r+1)}^{\text{ext}} \neq \emptyset$, there is $h_0 \in B_\alpha(1)$ with $v_{h_0} = v_0(\chi) - \gamma_1 r$. So for this h_0 the geodesic $[a(1, \xi), a(h_0, \xi)]$ runs through $v_0(\chi) - \gamma_1 r$. Since, for all $h \in B_\alpha(1)$, the vertex v_h is contained in $[a(1, \xi), v_{h_0}]$, it follows that $[a(h, \xi), v_0(\chi) - \gamma_1 r] \subset L_0$ holds for all h . Assume $h_1 m_\epsilon \notin [a(h_1, \xi), v_0(\chi)]$. Then we must have $[a(1, \xi), v_0(\chi) + 1] \subset L_0$ - a contradiction to the definition of $v_0(\chi)$, since $\chi \geq \Upsilon_0$. Thus, $h_1 m_\epsilon \in [a(h_1, \xi), v_0(\chi)]$, and hence $d_T(a(h_1, \xi), v_0(\chi) - \gamma_1 r) \geq k$. Combining the facts that $[a(h_1, \xi), v_0(\chi)]$ is $\Upsilon_1(\chi)$ -small (by Lemma 8.9) and that $r + 1 \leq k$ with our assumption that the measurement at $v_0(\chi)$ on γ_{h_1} is $\Upsilon_1(\chi)$ -small, it follows that the segment $[v_0(\chi) - \gamma_1 r, v_0(\chi) + \gamma_1 1]$ is contained in $L_k(\Upsilon_1(\chi))$ (cf. Figure 7 and Choosing Constants 8.5). Note that, if $r = k - 1$, then we indeed need all k steps in the iteration process used to build $L_k(\Upsilon_1(\chi))$ from L_0 in order to add this whole segment.

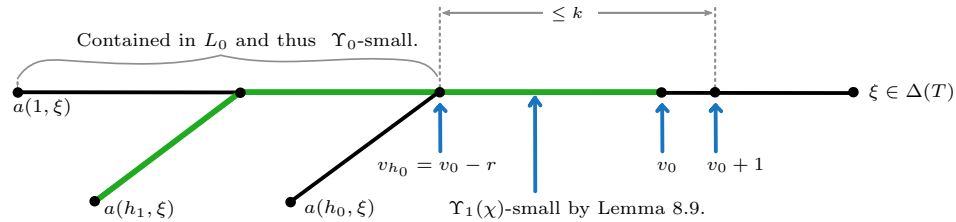


FIGURE 7. The segment $[v_0 - \gamma_1 r, v_0 + \gamma_1 1]$, with $v_0 = v_0(\chi)$, is contained in $L_k(\Upsilon_1(\chi))$.

Now recall that $\Upsilon_2(\chi) = \Upsilon_1(\Upsilon_1(\chi))$. Hence, $[a(h, \xi), v_0(\chi) + 1] \subset L_k(\Upsilon_1(\chi))$ is $\Upsilon_2(\chi)$ -small for all $h \in B_\alpha(1)$. In particular, the vertex $v_0(\Upsilon_2(\chi))$ has to lie on

$(v_0(\chi), \xi]$ and subsequently $s := d_T(v_0(\chi), v_0(\Upsilon_2(\chi)))$ is ≥ 1 . Moreover, it is

$$\begin{aligned} B_{v_0(\Upsilon_2(\chi))-\gamma_1(r+s)}^{\text{ext}} &= B_{v_0(\chi)-\gamma_1 r}^{\text{ext}} = \emptyset \quad \text{and} \\ B_{v_0(\Upsilon_2(\chi))-\gamma_1(r+s+1)}^{\text{ext}} &= B_{v_0(\chi)-\gamma_1(r+1)}^{\text{ext}} \neq \emptyset. \end{aligned} \quad \square$$

We are now ready to prove Proposition 8.1.

Proof of Proposition 8.1. Set $N := k + 1$ and let $\alpha > 0$ be given. Further define the following constants depending on α :

- Let Υ_0 be as in Choosing Constants 8.5.
- For $0 \leq i \leq k$ define $\Theta_i := \Upsilon_{2i+1}(\Upsilon_0)$ (also cf. Choosing Constants 8.5).
- Choose Θ_{-1} such that $\Theta_{-1} > \Upsilon_{2(k+1)}(\Upsilon_0)$. Note that the iterative construction of $\Upsilon_{2(k+1)}(\Upsilon_0)$ guarantees $\Theta_{-1} \geq \Upsilon_i(\Upsilon_0)$ for $1 \leq i \leq 2k + 1$ and, therefore, also $\Theta_{-1} > \Upsilon_0$.
- Define $\Theta_{k+1} := \Upsilon_1(\Theta_{-1})$.

By Lemma 8.4, $\mathcal{W}_\alpha := \bigcup_{i=0}^{k+1} \mathcal{W}_{\Theta_i}$ is a G -invariant collection of open \mathcal{F}_T -subsets of $G \times \Delta(T)$ and has order at most $k + 1$. It remains to verify the wideness claim. That is, for each $(1, \xi) \in G \times \Delta(T)$ we either have to find $0 \leq i \leq k + 1$ such that $B_\alpha(1) \times \{\xi\} \subseteq W(v, \Theta_i)$ for some $v \in V(T)$ or conclude that $[a(1, \xi), \xi]$ is Θ_{-1} -small and ξ lies in ∂T .

Let $(1, \xi) \in G \times \Delta(T)$ and recall the notation $\gamma_h, B_v^{\text{inn}}$ and B_v^{ext} from Notation 8.6. **Case 1:** Assume that γ_1 is Θ_{-1} -small. If $\xi \in \partial T$, there is nothing to show. If $\xi \in V_\infty(T)$, then by Lemma 8.9 (for $x = \xi$ and $\chi = \Theta_{-1}$) and the definition of $W(\xi, \Upsilon_1(\Theta_{-1}))$ it follows that

$$B_\alpha(1) \times \{\xi\} \subset W(\xi, \Upsilon_1(\Theta_{-1})) = W(\xi, \Theta_{k+1}).$$

Case 2: Assume that γ_1 is not Θ_{-1} -small. In particular, γ_1 is not Υ_0 -small and $v_0(\Upsilon_0)$ exists. Thus, by Lemma 8.7, $B_{v_0(\Upsilon_0)}^{\text{ext}} = \emptyset$ and there is a unique $r_0 \geq 0$ such that

$$B_{v_0(\Upsilon_0)-\gamma_1 r_0}^{\text{ext}} = \emptyset, \text{ but } B_{v_0(\Upsilon_0)-\gamma_1(r_0+1)}^{\text{ext}} \neq \emptyset.$$

If $r_0 \geq k$, we can apply Lemma 8.10 with $\chi = \Upsilon_0$. Since γ_1 is not $\Upsilon_1(\Upsilon_0)$ -small, it either follows that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_0), \Upsilon_1(\Upsilon_0)) = W(v_0(\Upsilon_0), \Theta_0)$$

or that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_1(\Upsilon_0)), \Upsilon_1(\Upsilon_0)) = W(v_0(\Upsilon_1(\Upsilon_0)), \Theta_0).$$

If $r_0 \leq k - 1$, we can apply Lemma 8.11 with $r = r_0$ and $\chi = \Upsilon_0$. Since γ_1 is not $\Upsilon_2(\Upsilon_0)$ -small, it either follows that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_0), \Upsilon_1(\Upsilon_0)) = W(v_0(\Upsilon_0), \Theta_0)$$

or that $v_0(\Upsilon_2(\Upsilon_0))$ exists and that there is an $s_0 \geq 1$ such that

$$B_{v_0(\Upsilon_2(\Upsilon_0))-\gamma_1(r_0+s_0)}^{\text{ext}} = \emptyset, \text{ but } B_{v_0(\Upsilon_2(\Upsilon_0))-\gamma_1(r_0+s_0+1)}^{\text{ext}} \neq \emptyset.$$

This process can be iterated for $0 \leq i \leq k$. Assume that we have already constructed an $r_i \geq 0$ with

$$B_{v_0(\Upsilon_{2i}(\Upsilon_0))-\gamma_1 r_i}^{\text{ext}} = \emptyset, \text{ but } B_{v_0(\Upsilon_{2i}(\Upsilon_0))-\gamma_1(r_i+1)}^{\text{ext}} \neq \emptyset,$$

where for $i = 0$ we interpret $\Upsilon_0(\Upsilon_0)$ as Υ_0 . If $r_i \geq k$, we can apply Lemma 8.10 with $\chi = \Upsilon_{2i}(\Upsilon_0)$. Since γ_1 is not $\Upsilon_{2i+1}(\Upsilon_0)$ -small (as long as $i \leq k$), it either follows that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_{2i}(\Upsilon_0)), \Upsilon_{2i+1}(\Upsilon_0))$$

or that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_{2i+1}(\Upsilon_0)), \Upsilon_{2i+1}(\Upsilon_0)).$$

If $r_i \leq k - 1$, we can apply Lemma 8.11 with $r = r_i$ and $\chi = \Upsilon_{2i}(\Upsilon_0)$. Since γ_1 is not $\Upsilon_{2(i+1)}(\Upsilon_0)$ -small (as long as $i \leq k$), it either follows that

$$B_\alpha(1) \times \{\xi\} \subset W(v_0(\Upsilon_{2i}(\Upsilon_0)), \Upsilon_{2i+1}(\Upsilon_0))$$

or that $v_0(\Upsilon_{2(i+1)}(\Upsilon_0))$ exists and that there is $s_i \geq 1$ such that

$$B_{v_0(\Upsilon_{2(i+1)}(\Upsilon_0))-\gamma_1}^{\text{ext}}(r_i + s_i) = \emptyset, \text{ but } B_{v_0(\Upsilon_{2(i+1)}(\Upsilon_0))-\gamma_1}^{\text{ext}}(r_i + s_i + 1) \neq \emptyset.$$

In this case we define $r_{i+1} := r_i + s_i$ and repeat the process. Since in each step $s_i \geq 1$, there is $0 \leq j \leq k$ such that $r_j \geq k$. Hence, all constants $\Upsilon_{2i+1}(\Upsilon_0)$ considered in the process are among $\Theta_0, \dots, \Theta_k$. \square

This concludes the construction of suitable G -invariant collections of open \mathcal{F} -subsets that are wide for $G \times \Delta(T) \setminus G \times_{\Theta_{-1}} \partial T$ (with Θ_{-1} depending on $\alpha > 0$).

9. THE SEGMENT-FLOW SPACE FS_Θ

The aim of this and the following section is to construct—for all Θ —a G -invariant collection of open \mathcal{F} -subsets that is wide for $G \times_\Theta \partial T$. Our construction will be analogously to the one in [Bar17] for relatively hyperbolic groups. For this, we first define another proper G -invariant metric on $E^k(T)$ which essentially coincides with the path-metric d_T when restricted to segments lying on a small geodesic.

From now on assume that k is even and recall that in this case the midpoint $m(\sigma)$ of any $\sigma \in E^k(T)$ is a vertex of T .

Definition 9.1. Let $\Theta > 0$ and let $\sigma, \sigma' \in E^k(T)$. The geodesic $\gamma = [m(\sigma), m(\sigma')]$ is called *extended Θ -small*, if the following holds:

- γ is Θ -small (as in Definition 7.2).
- If $d_T(m(\sigma), m(\sigma')) \geq k$, then

$$d_{E^k}(\sigma, \sigma_{o(\gamma)+\gamma k}) \leq \Theta$$

and

$$d_{E^k}(\sigma_{t(\gamma)}, \sigma') \leq \Theta.$$

- If $d_T(m(\sigma), m(\sigma')) \leq k - 1$, then $d_{E^k}(\sigma, \sigma') \leq \Theta$.

Remember that the $+\gamma$ -operation on vertices of γ is meant as introduced in Section 7.

Definition 9.2. An *oriented segment (of length k)* in T is an element $\sigma \in E^k(T)$ together with a vertex $b \in \{o(\sigma), t(\sigma)\}$. For $\sigma, \sigma' \in E^k(T)$ denote by $\mathcal{M}_\Theta(\sigma, \sigma')$ the set of all sequences $(\sigma_0, b_0), (\sigma_1, b_1), \dots, (\sigma_n, b_n)$ of oriented segments of length k in T such that $\sigma = \sigma_0$ and $\sigma' = \sigma_n$ (as unoriented segments) and the geodesics $[m(\sigma_i), m(\sigma_{i+1})]$ are extended Θ -small. For brevity, elements of $\mathcal{M}_\Theta(\sigma, \sigma')$ are

called *admissible sequences* (for the pair (σ, σ')) and the vertices b_i are dropped from the notation. Finally, we define

$$d_\Theta(\sigma, \sigma') := \min_{\mathcal{M}_\Theta(\sigma, \sigma')} \sum_{i=0}^{n-1} (d(m(\sigma_i), m(\sigma_{i+1})) + d(o(\sigma_i), o(\sigma_{i+1})) + d(t(\sigma_i), t(\sigma_{i+1}))),$$

where $d := d_T$ denotes the path-metric on T .

Lemma 9.3. *d_Θ is a proper G -invariant generalised metric on $E^k(T)$. Furthermore, if $\sigma, \sigma' \in E^k(T)$ both lie on the same Θ -small geodesic, then $d_\Theta(\sigma, \sigma') = 3 \cdot d_T(m(\sigma), m(\sigma'))$.*

Proof. The G -invariance of d_Θ follows from the G -invariance of both d_T and d_{E^k} . Symmetry follows from $\mathcal{M}_\Theta(\sigma, \sigma') = \mathcal{M}_\Theta(\sigma', \sigma)$. The triangle inequality holds, since we can concatenate an admissible sequence realising $d_\Theta(\sigma, \sigma')$ with an admissible sequence realising $d_\Theta(\sigma', \sigma'')$ to obtain an admissible sequence for the pair (σ, σ'') . Taking mid-, start- and endpoints of segments into account for the definition of $d_\Theta(\sigma, \sigma')$ guarantees that d_Θ is indeed a generalised metric (instead of a generalised pseudo-metric).

For two elements σ and σ' in $E^k(T)$ lying on the same Θ -small geodesic, the sequence $(\sigma, o(\sigma)), (\sigma', o(\sigma'))$ is an admissible sequence and yields $d_\Theta(\sigma, \sigma') \leq 3 \cdot d_T(m(\sigma), m(\sigma'))$. Combined use of the triangle inequality for d_T and the fact that σ, σ' lie on the same Θ -small geodesic gives that the admissible sequence $(\sigma, o(\sigma)), (\sigma', o(\sigma'))$ already realises $d_\Theta(\sigma, \sigma')$. It remains to show that d_Θ is proper:

Let $R > 0$ and $\sigma \in E^k(T)$ be given. Note that, since the distance between two vertices (with respect to d_T) is an integer, if $\sigma \neq \sigma'$, then $d_\Theta(\sigma, \sigma') \geq 1$. So, if $\sigma' \in B_R^{d_\Theta}(\sigma)$, then there is an admissible sequence $(\sigma_0, b_0), (\sigma_1, b_1), \dots, (\sigma_r, b_r)$, $r \leq R$ for the pair (σ, σ') such that $d_T(m(\sigma_i), m(\sigma_{i+1})) \leq R$ for all $0 \leq i \leq r-1$.

Since d_{E^k} is proper, given σ_i there are only finitely many segments $\sigma \in E^k(T)$ with $d_{E^k}(\sigma_i, \sigma) \leq \Theta$. Thus, since $[m(\sigma_i), m(\sigma_{i+1})]$ has to be extended Θ -small and of length $\leq R$ (with respect to d_T), there are only finitely many possibilities for σ_1 . And given any of those possible σ_1 , there are only finitely many possibilities for σ_2 . Iterating this argument R times we conclude that there are only finitely many possibilities for $\sigma_r = \sigma'$. \square

Next we will define a segment-flow space FS_Θ as a certain subset of the space $E^k(T) \times \Delta_+(T) \times \Delta_+(T)$. The definition given here is a slight adaptation of the coarse Θ -flow space in [Bar17, Definition 3.4] to our situation on trees.

Definition 9.4. Let $\Theta > 0$. Let $Z := \Delta_+(T) \times \Delta_+(T)$ be equipped with the subspace topology. Furthermore, let

$$FS_\Theta := \{(\sigma, \xi_-, \xi_+) \in E^k(T) \times Z \mid \sigma \subset [\xi_-, \xi_+] \text{ and } [\xi_-, \xi_+] \text{ is } \Theta\text{-small}\},$$

where the notion of a Θ -small geodesic is as in Definition 7.2.

Before we list properties of FS_Θ , we cite a definition of Bartels, which we will need briefly while applying [Bar17, Theorem 1.1] to obtain long thin covers for FS_Θ .

Definition 9.5. (see [Bar17, p. 750, before Theorem 1.1]) Let (V, d) be a generalised discrete metric space. Let $\alpha \geq 0$. A subset $S \subset V$ is α -separated if $d(x, y) \geq \alpha$ for any two distinct points $x, y \in S$. A subspace $V_0 \subset V$ has the (D, R) -doubling property if for all $\alpha \geq R$ the following holds: if $S \subset V_0$ is α -separated and contained in a 2α -ball, then the cardinality of S is at most D .

The following lemma collects the properties of $E^k(T)$, Z and FS_Θ needed to apply [Bar17, Theorem 1.1] later on.

Lemma 9.6. *Let $\Theta > 0$ and let d_Θ be as in Definition 9.2. Let Z and FS_Θ be as above. Then*

- a) $(E^k(T), d_\Theta)$ is a discrete countable proper generalised metric space with a proper isometric G -action.
- b) Z is a separable metric space with an action of G by homeomorphisms. FS_Θ is separable and metrisable as well.
- c) FS_Θ is a closed G -invariant subspace of $E^k(T) \times Z$ (equipped with the diagonal action of G).
- d) The (small inductive) dimension of FS_Θ is 0.
- e) There are $D, R > 0$, independent of Θ , such that for all $(\xi_-, \xi_+) \in Z$ the subspace $E^k(T)_{(\xi_-, \xi_+)} := \{\sigma \in E^k(T) \mid (\sigma, \xi_-, \xi_+) \in FS_\Theta\}$ of $E^k(T)$ has the (D, R) -doubling property.
- f) For all $(\sigma, \xi_-, \xi_+) \in FS_\Theta$, the isotropy group $G_{(\xi_-, \xi_+)} = G_{\xi_-} \cap G_{\xi_+}$ belongs to the family $\mathcal{F}_\partial \cup \mathcal{F}_{(k)}$, where $\mathcal{F}_{(k)} := \{H \leq G \mid \exists \sigma \in E^k(T) : H \leq G_{o(\sigma)} \cap G_{t(\sigma)}\} \subset \mathcal{FIN}(G)$. In particular, these isotropy groups all belong to $\mathcal{F} = \mathcal{F}_T \cup \mathcal{F}_\partial$.

Proof. a) is immediate since d_Θ only takes values in \mathbb{N} and the action of G on $E^k(T)$ has finite point stabilisers.

b) Since we assumed T to be countable, $\Delta_+(T)$ is separable and metrisable by Lemma 2.4. So Z is as well. Since $E^k(T)$ is discrete and countable, $E^k(T) \times Z$ is separable and metrisable and hence is FS_Θ .

d) By the product theorem [Nag65, Theorem II.5, p. 20]³ and Lemma 2.8 the space $Z = \Delta_+(T) \times \Delta_+(T)$ has dimension 0. Since a discrete space V has dimension 0, $E^k(T) \times Z$ has dimension 0 by [Nag65, Theorem II.5, p. 20]. Finally, by [Nag65, Theorem II.3, p. 19], the dimension does not increase when taking subspaces and hence FS_Θ has dimension 0.

f) Since $G_{(\xi_-, \xi_+)} = G_{\xi_-} \cap G_{\xi_+}$, if $(\xi_-, \xi_+) \in \partial T \times \partial T \setminus \text{diag}$, then $G_{(\xi_-, \xi_+)} \in \mathcal{F}_\partial$ by definition. If at least one of the points ξ_-, ξ_+ lies in T , then, by definition of FS_Θ , the group $G_{\xi_-} \cap G_{\xi_+}$ is contained in the pointwise stabiliser G_σ of some $\sigma \in E^k(T)$, and hence is finite. It remains to show c) and e).

c) Since $E^k(T) \times Z$ is metrisable, we can work with sequences. Moreover, without loss of generality any convergent sequence has the form

$$(\sigma, \xi_{-,n}, \xi_{+,n}) \rightarrow (\sigma, \xi_-, \xi_+) \in E^k(T) \times Z,$$

since $E^k(T)$ is discrete. Now, if all $(\sigma, \xi_{-,n}, \xi_{+,n})$ lie in FS_Θ , then $\sigma \subset [\xi_{-,n}, \xi_{+,n}]$ holds for all $n \in \mathbb{N}$. Hence, the topology of $\Delta_+(T)$ forces σ to lie on $[\xi_-, \xi_+]$ (though $\sigma = [\xi_-, \xi_+]$ is possible). The geodesic $[\xi_-, \xi_+]$ is Θ -small, since the geodesics $[\xi_{-,n}, \xi_{+,n}]$ are and any finite subsegment of $[\xi_-, \xi_+]$ is eventually contained in some $[\xi_{-,N}, \xi_{+,N}]$. G -invariance of FS_Θ is immediate, since the notion of a Θ -small geodesic is G -invariant.

e) We will show that for all (ξ_-, ξ_+) the subspace $E^k(T)_{(\xi_-, \xi_+)}$ has the $(5, 3)$ -doubling property: So let $\alpha \geq R = 3$. First, note that $E^k(T)_{(\xi_-, \xi_+)}$ is empty if the

³This and the following results referenced are formulated for the strong inductive dimension. However, recall that for separable metrisable spaces, the small inductive dimension coincides with the covering dimension (and the strong inductive dimension) [Nag65, Theorem IV.1, p. 90].

geodesic $[\xi_-, \xi_+]$ is not Θ -small or is of length $\leq k - 1$. Otherwise $E^k(T)_{(\xi_-, \xi_+)}$ is equal to $E^k(T) \cap [\xi_-, \xi_+]$. In this case, we can take midpoints of segments to obtain a bijection between $E^k(T)_{(\xi_-, \xi_+)}$ and (a subset of) \mathbb{Z} . Since $d_\Theta(\sigma, \sigma') = 3 \cdot d_T(m(\sigma), m(\sigma'))$ for all σ, σ' on $[\xi_-, \xi_+]$, any set of the form $B_{2\alpha}(\tilde{\sigma}) \cap E^k(T)_{(\xi_-, \xi_+)}$ consists of at most $\frac{4}{3}\alpha + 1$ elements of $E^k(T)$. Furthermore, if $\sigma, \sigma' \in E^k(T)_{(\xi_-, \xi_+)}$ belong to an α -separated subset, then $d_T(m(\sigma), m(\sigma')) \geq \frac{\alpha}{3}$ follows from $d_\Theta(\sigma, \sigma') = 3 \cdot d_T(m(\sigma), m(\sigma'))$. Thus, there are at most $\frac{3}{\alpha} \cdot (\frac{4}{3}\alpha + 1) \leq 5 =: D$ elements in any α -separated subset of $E^k(T)_{(\xi_-, \xi_+)}$ which is contained in a 2α -ball. \square

Next, we define a family of maps from $G \times_\Theta \partial T$ to FS_Θ that formalise the idea of letting an element σ of $E^k(T)$ ‘flow along a Θ -small geodesic $[a(g, \xi), \xi]$ ’. Morally, this family of maps could be thought of as a partial flow on FS_Θ .

Definition 9.7. Let $\tau \in \mathbb{N}$ and $\Theta' \geq \Theta > \frac{k}{2}$. We define

$$\phi_{\Theta, \Theta', \tau} : G \times_\Theta \partial T \longrightarrow FS_{\Theta'}, \quad (g, \xi) \mapsto (\sigma_\tau(g, \xi), a(g, \xi), \xi),$$

where $\sigma_\tau(g, \xi)$ is the unique geodesic segment of length k on $[a(g, \xi), \xi]$ such that

$$d_T(a(g, \xi), m(\sigma_\tau(g, \xi))) = \tau.$$

Lemma 9.8. $\phi_{\Theta, \Theta', \tau}$ is continuous and G -equivariant.

Proof. G -equivariance is clear, since the action of G on T is simplicial. Let $(g, \xi_n) \in G \times_\Theta \partial T$ be a sequence converging to some $(g, \xi) \in G \times_\Theta \partial T$. Since $a(g, \eta) \in \{gw_0, gw'_0\}$ for arbitrary η , we can assume without loss of generality that $a(g, \xi_n) = a(g, \xi)$ for all $n \in \mathbb{N}$. The convergence of $\xi_n \rightarrow \xi$ then already implies that $\sigma_\tau(g, \xi_n) = \sigma_\tau(g, \xi)$ for all but finitely many n . \square

10. COVERS FOR $G \times_\Theta \partial T$

Given some threshold $\Theta > 0$, this section provides an open G -invariant $\mathcal{F}_\partial \cup \mathcal{F}_{(k)}$ -cover of $G \times_\Theta \partial T$ that is wide for $G \times_\Theta \partial T$. We start by using [Bar17, Theorem 1.1] to obtain a family of covers for the flow space $FS_{\Theta'}$ for any Θ' .

Corollary 10.1. *Let $\Theta' > 0$. Let $FS_{\Theta'}$ be as before and let D and R be as in Lemma 9.6. Then there is $N \in \mathbb{N}$ independent of Θ' , such that for any $\alpha > 0$ there is an open G -invariant $\mathcal{F}_\partial \cup \mathcal{F}_{(k)}$ -cover $\mathcal{W}_{\Theta', \alpha}$ of $FS_{\Theta'}$ satisfying*

- a) $\dim \mathcal{W}_{\Theta', \alpha} \leq N$,
- b) for every $(\sigma, \xi_-, \xi_+) \in FS_{\Theta'}$ there is $W \in \mathcal{W}_{\Theta', \alpha}$ such that

$$(B_\alpha^{d_{\Theta'}}(\sigma) \times \{(\xi_-, \xi_+)\}) \cap FS_{\Theta'} \subset W.$$

Proof. This is the conclusion of [Bar17, Theorem 1.1]. That the assumptions of this theorem are satisfied by $FS_{\Theta'}$ follows from Lemma 9.6. \square

Pulling back these covers along $\phi_{\Theta, \Theta', \tau}$ gives the following corollary.

Corollary 10.2. *There is N such that for all $\alpha > 0, \Theta' \geq \Theta > \frac{k}{2}$ and $\tau \in \mathbb{N}$ the set*

$$\phi_{\Theta, \Theta', \tau}^{-1}(\mathcal{W}_{\Theta', \alpha}) := \{\phi_{\Theta, \Theta', \tau}^{-1}(W) \mid W \in \mathcal{W}_{\Theta', \alpha}\}$$

is an open G -invariant $\mathcal{F}_\partial \cup \mathcal{F}_{(k)}$ -cover of $G \times_\Theta \partial T$ of dimension $\leq N$.

Before we show that among these covers we can find one which is wide for $G \times_\Theta \partial T$, we will state two technical assertions.

Lemma 10.3. *Let $\alpha > 0$. Let $v \in V(T)$ and $g_\tau v \rightarrow \xi \in \partial T$ for $\tau \rightarrow \infty$. Then, for all $h \in B_\alpha(1)$, the sequence $g_\tau h v$ also converges to ξ .*

Proof. Fix a basepoint $b \in T$. So $[b, g_\tau v]$ converges uniformly on compacta to $[b, \xi]$. Let $C := \max\{d_T(v, hv) \mid h \in B_\alpha(1)\}$. Since $d_T(g_\tau v, g_\tau hv) = d_T(v, hv) \leq C$ for all τ , the sequence $[b, g_h \tau v]$ converges uniformly on compacta to $[b, \xi]$ if and only if the sequence $[b, g_\tau v]$ does. \square

Lemma 10.4. *Let $\Theta > 0$. Let $\xi_{-,n} \rightarrow \xi_-$ and $\xi_{+,n} \rightarrow \xi_+$ be convergent sequences in $\Delta_+(T)$. Furthermore, assume that there is a segment $\sigma \in E^k(T)$ such that σ lies on all geodesics $[\xi_{-,n}, \xi_{+,n}]$ and that for all $r > 0$ the set $B_r^{d_T}(\sigma) := \{z \in T \mid d_T(z, \sigma) \leq r\}$ contains only finitely many of the $\xi_{-,n}, \xi_{+,n}$. If the geodesics $[\xi_{-,n}, \xi_{+,n}]$ are Θ -small, then $\xi_-, \xi_+ \in \partial T$ and $[\xi_-, \xi_+]$ is Θ -small.*

Proof. Assume that $\xi_+ \notin \partial T$. If $\xi_{+,n} = \xi_+$ for all $n \geq n_0$, there is—in contradiction to the assumptions of the lemma—some $r > 0$ such that $B_r^{d_T}(\sigma)$ contains infinitely many of $\xi_{+,n}$. So, for $\xi_{+,n} \rightarrow \xi_+ \in V(T)$ to hold, the union of the $[\xi_+, \xi_{+,n}]$ must contain infinitely many of the edges incident to ξ_+ . Since d_{E^k} is proper, this contradicts the other assumption of the lemma, that all $[o(\sigma), \xi_{+,n}]$ are Θ -small. The argument for $\xi_- \in \partial T$ is the same and the geodesic $[\xi_-, \xi_+]$ is Θ -small since all geodesics $[\xi_{-,n}, \xi_{+,n}]$ are. \square

Proposition 10.5. *Retain the assumptions of Proposition 4.1. Then there is $N \in \mathbb{N}$ such that for all $\alpha > 0$ and all $\Theta > 0$ there is a G -invariant collection $\mathcal{U}_{\Theta, \alpha}$ of open $\mathcal{F}_\partial \cup \mathcal{F}_{(k)}$ -subsets of $G \times \Delta(T)$ such that the order of $\mathcal{U}_{\Theta, \alpha}$ is at most N and $\mathcal{U}_{\Theta, \alpha}$ is wide for $G \times_\Theta \partial T$.*

Proof. (following the proof of [Bar17, Proposition 3.2]) Let N be as in Corollary 10.1 and Corollary 10.2. Let $\alpha, \Theta > 0$ be given. Recall that L_0 is the finite subtree of T spanned by the vertices $\{hw_0, hw'_0 \mid h \in B_\alpha(1)\}$. Let $C = C(\alpha)$ be the diameter of L_0 with respect to d_T . (In particular, for any $\xi \in \Delta_+(T)$, $h \in B_\alpha(1)$ and $g \in G$ the distance of $a(g, \xi)$ to $a(gh, \xi)$ is bounded by C .) Set $\alpha' := 3C$ and $\Theta' := \Upsilon_1(\Theta)$ (cf. Choosing Constants 8.5). Finally, let $\mathcal{W}_{\Theta', \alpha'}$ be as in Corollary 10.1.

We will show that there is $\tau \in \mathbb{N}$ such that $\phi_{\Theta, \Theta', \tau}^{-1}(\mathcal{W}_{\Theta', \alpha'})$ is (α) -wide for $G \times_\Theta \partial T$. Once we have proven this, we can thicken $\phi_{\Theta, \Theta', \tau}^{-1}(\mathcal{W}_{\Theta', \alpha'})$ to a G -invariant collection $\mathcal{U}_{\Theta, \alpha}$ of open \mathcal{F} -subsets of $G \times \Delta(T)$ that is still (α) -wide for $G \times_\Theta \partial T$ (see [Bar17, Appendix B], [BL12b, Lemma 4.14]). Since all $\phi_{\Theta, \Theta', \tau}^{-1}(\mathcal{W}_{\Theta', \alpha'})$ have dimension at most N by Corollary 10.2, the collection $\mathcal{U}_{\Theta, \alpha}$ will have order $\leq N$ as well as the thickening process does not increase the order [Bar17, Lemma B.2].

To show that for the given $\alpha, \Theta > 0$ there is a $\phi_{\Theta, \Theta', \tau}^{-1}(\mathcal{W}_{\Theta', \alpha'})$ that is (α) -wide, start by assuming the contrary.

Assumption: For all $\tau \in \mathbb{N}$ there is $(g_\tau, \xi_\tau) \in G \times_\Theta \partial T$ such that for all $W \in \mathcal{W}_{\Theta', \alpha'}$

$$B_\alpha(g_\tau) \times \{\xi_\tau\} \not\subset \phi_{\Theta, \Theta', \tau}^{-1}(W) .$$

Applying $\phi_{\Theta, \Theta', \tau}$ to (g_τ, ξ_τ) we obtain a segment $\sigma_\tau(g_\tau, \xi_\tau)$ of length k on $[a(g_\tau, \xi_\tau), \xi_\tau]$. Let σ_1 be the first segment of length k on $[w_0, w'_0]$ and σ_2 be the last segment of length k on $[w_0, w'_0]$ (those segments exist since $[w_0, w'_0]$ has length $\geq 5k$). Since $[a(g_\tau, \xi_\tau), \xi_\tau]$ contains $[a(g_\tau, \xi_\tau), gm_\epsilon]$, the first segment of length k on $[a(g_\tau, \xi_\tau), \xi_\tau]$ is either $g_\tau \sigma_1$ or $g_\tau \sigma_2$. Furthermore, since all $[a(g_\tau, \xi_\tau), \xi_\tau]$ are Θ -small and their respective first segments of length k lie in at most two different G -orbits of $G \curvearrowright E^k(T)$, by our special construction of d_{E^k} (cf. Lemma 7.1), it follows that the segments $\sigma_\tau(g_\tau, \xi_\tau)$ can only belong to a finite number of G -orbits of $G \curvearrowright E^k(T)$.

Since $\phi_{\Theta, \Theta', \tau}^{-1}(\mathcal{W}_{\Theta', \alpha'})$ is a G -invariant collection we can assume (after passing to a subsequence) that there is a segment $\sigma \in E^k(T)$ such that $\sigma = \sigma_\tau(g_\tau, \xi_\tau)$ for all τ .

Since $\Delta_+(T)$ is a closed subspace of \overline{T}^{obs} , it is compact as well and by passing to a further subsequence (twice) we can enforce the existence of $\xi_+ = \lim_\tau \xi_\tau$ and $\xi_- = \lim_\tau a(g_\tau, \xi_\tau)$ in $\Delta_+(T)$.

Since τ increases, no $B_\tau^{d_T}(\sigma)$ contains infinitely many of the $a(g_\tau, \xi_\tau)$. Moreover, by assumption, all ξ_τ lie in ∂T . Hence, by Lemma 10.4, we obtain that ξ_- and ξ_+ both lie in ∂T and that $[\xi_-, \xi_+]$ is Θ -small. Furthermore, σ lies on $[\xi_-, \xi_+]$. In other words, $(\sigma, \xi_-, \xi_+) \in FS_\Theta \subseteq FS_{\Theta'}$.

Now, by Corollary 10.1, there is $W_0 \in \mathcal{W}_{\Theta', \alpha'}$ such that

$$(B_{\alpha'}^{d_{\Theta'}}(\sigma) \times \{(\xi_-, \xi_+)\}) \cap FS_{\Theta'} \subset W_0.$$

As $d_{\Theta'}$ is proper, the ball $B_{\alpha'}^{d_{\Theta'}}(\sigma)$ is finite and so is $M := B_{\alpha'}^{d_{\Theta'}}(\sigma) \cap [\xi_-, \xi_+]$. Since $\xi_-, \xi_+ \in \partial T$ and W_0 is open, there are disjoint open neighbourhoods U_- and U_+ in $\Delta_+(T)$ of ξ_- and ξ_+ , respectively, such that

$$(M \times U_- \times U_+) \cap FS_\Theta \subset W_0.$$

Without loss of generality we can assume that U_+ is of the form $M(\xi_+, a)$ (cf. Definition 2.2) for some $a \in T$ with $d_T(t(\sigma), a) > C$ (the purpose of this technical assumption will only be clear at the very end of this proof).

By Lemma 10.3, we still have $\lim_\tau a(g_\tau h, \xi_\tau) = \xi_-$ for all $h \in B_\alpha(1)$. So we can find $\tau_{0,h}$ such that $a(g_\tau h, \xi_\tau) \in U_-$ and $\xi_\tau \in U_+$ for all $\tau \geq \tau_{0,h}$. Since $B_\alpha(1)$ is finite and $B_\alpha(g_\tau) = g_\tau B_\alpha(1)$, there is a τ_0 such that $a(g, \xi_\tau) \in U_-$ and $\xi_\tau \in U_+$ for all $\tau \geq \tau_0$ and $g \in B_\alpha(g_\tau)$. Since M is finite as well, we can (if necessary) enlarge τ_0 further to obtain that all elements of M already lie on $[a(g_{\tau_0}, \xi_{\tau_0}), \xi_{\tau_0}]$. We now claim that there is some $\tau \geq \tau_0$ such that

$$g_\tau B_\alpha(1) \times \{\xi_\tau\} = B_\alpha(g_\tau) \times \{\xi_\tau\} \subset \phi_{\Theta, \Theta', \tau}^{-1}(W_0),$$

which would be a contradiction to the assumption we started with. To prove this claim it is sufficient to find τ such that

$$(\sigma_\tau(g_\tau h, \xi_\tau), a(g_\tau h, \xi_\tau), \xi_\tau) \in (M \times U_- \times U_+) \cap FS_{\Theta'}$$

for all $h \in B_\alpha(1)$. $(\sigma_\tau(g_\tau h, \xi_\tau), a(g_\tau h, \xi_\tau), \xi_\tau) \in FS_{\Theta'}$ holds by Lemma 8.9 and Lemma 8.8. By the way τ_0 was defined, it is guaranteed that $\xi_\tau \in U_+$ and $a(g_\tau h, \xi_\tau) \in U_-$ for all $\tau \geq \tau_0$. So it only remains to find $\tau \geq \tau_0$ large enough such that $\sigma_\tau(g_\tau h, \xi_\tau) \in M$ holds for all $h \in B_\alpha(1)$.

Let v_h be the first vertex on $[a(g_\tau h, \xi_\tau), \xi_\tau]$ that also lies on $[a(g_\tau, \xi_\tau), \xi_\tau]$, i.e.

$$[a(g_\tau h, \xi_\tau), \xi_\tau] \cap [a(g_\tau, \xi_\tau), \xi_\tau] = [v_h, \xi_\tau].$$

Since $B_\alpha(1)$ is finite, there is a vertex $v = v_{h_0}$ which—among the v_h —has maximal distance to $a(g_\tau, \xi_\tau)$. In particular, $[v, \xi_\tau]$ is contained in $[a(g_\tau h, \xi_\tau), \xi_\tau]$ for all $h \in B_\alpha(1)$. Note that, since all $a(g_\tau h, \xi_\tau)$ lie in U_- , all v_h lie in U_- as well. In particular, v lies in U_- . Moreover, the vertex v lies on the geodesic $[a(g_\tau h_0, \xi_\tau), a(g_\tau h, \xi_\tau)]$ for all $h \in B_\alpha(1)$. Since the length of $[a(g_\tau h_0, \xi_\tau), a(g_\tau h, \xi_\tau)]$ is bounded by C , the vertex v has distance at most C to any of the $a(g_\tau h, \xi_\tau)$. Hence, (by choosing $\tau \geq \tau_0$ large enough) we can force $\sigma_\tau(g_\tau h, \xi_\tau)$ to lie on $[v, \xi_\tau]$ for all $h \in B_\alpha(1)$. Since $d_T(a(g_\tau, \xi_\tau), m(\sigma_\tau(g_\tau, \xi_\tau))) = \tau$, the bound C and the triangle inequality for d_T imply

$$\tau + C \leq d_T(a(g_\tau, \xi_\tau), m(\sigma_\tau(g_\tau h, \xi_\tau))) \geq \tau - C.$$

Both $\sigma_\tau(g_\tau h, \xi_\tau)$ and $\sigma_\tau(g_\tau, \xi_\tau)$ lie on $[v, \xi_\tau]$ and it follows that

$$d_T(m(\sigma_\tau(g_\tau h, \xi_\tau)), m(\sigma_\tau(g_\tau, \xi_\tau))) \leq C.$$

Since the geodesic $[a(g_\tau, \xi_\tau), \xi_\tau]$ is Θ - and hence Θ' -small and contains both the segments $\sigma_\tau(g_\tau h, \xi_\tau)$ and $\sigma_\tau(g_\tau, \xi_\tau)$, by definition of $d_{\Theta'}$ it follows that

$$d_{\Theta'}(\sigma_\tau(g_\tau h, \xi_\tau), \sigma_\tau(g_\tau, \xi_\tau)) = 3 \cdot d_T(m(\sigma_\tau(g_\tau h, \xi_\tau)), m(\sigma_\tau(g_\tau, \xi_\tau))) \leq 3C = \alpha'.$$

Hence, $\sigma_\tau(g_\tau h, \xi_\tau) \in B_{\alpha'}^{d_{\Theta'}}(\sigma_\tau(g_\tau, \xi_\tau))$ for all $h \in B_\alpha(1)$. Recall that $\sigma = \sigma_\tau(g_\tau, \xi_\tau)$ and $U_+ = M(\xi_+, a)$ where a satisfies $d_T(t(\sigma), a) \geq C$. Therefore, $\sigma_\tau(g_\tau h, \xi_\tau)$ indeed lies on $[\xi_-, \xi_+]$ and is contained in $B_{\alpha'}^{d_{\Theta'}}(\sigma)$, so it is contained in M . \square

This concludes the last technical part that was used in the proof of Theorem A.

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